Math 157 Honors Calculus I

Note 1, 9/6/2018

Let $\mathbb{Q}$ and $\mathbb{I}$ denote the set of rational and irrational numbers, respectively. Thus $x \in \mathbb{Q}$ means that the number x can be written as the fraction m/n for some integers m, n with $n \neq 0$, while $x \in \mathbb{I}$ means that no such representation of x as a fraction is possible.

Claim 1. *Suppose x and y are both rational. Then $x + y$ and xy are rational. If $y \neq 0$, then x/y is rational.*

This is straightforward. By the assumption we can write $x = m/n$ and $y = m'/n'$ for some integers m, m', n, n'. Then

$$x + y = \frac{mn' + m'n}{nn'} \qquad \text{and} \qquad xy = \frac{mm'}{nn'}.$$

Since $mn' + m'n$, mm' and nn' are all integers, it follows that $x + y \in \mathbb{Q}$ and $xy \in \mathbb{Q}$. If $y \neq 0$, we have $m' \neq 0$, so

$$\frac{x}{y} = \frac{mn'}{nm'}.$$

Since mn' and nm' are integers, we conclude that $x/y \in \mathbb{Q}$.

Claim 2. *Suppose x is rational and y is irrational. Then*

(i) *$x + y$ is irrational.*

(ii) *xy is irrational unless $x = 0$ (in which case $xy = 0$ is rational).*

(iii) *x/y is irrational unless $x = 0$ (in which case $x/y = 0$ is rational).*

To see (i), suppose $x + y$ were rational. Then by Claim 1, $-x = (-1)x \in \mathbb{Q}$ and so $y = -x + (x + y) \in \mathbb{Q}$, which would contradict our assumption. Hence $x + y \in \mathbb{I}$.

The argument for (ii) is similar. Let $x \neq 0$ and suppose xy were rational. Then by Claim 1, $1/x \in \mathbb{Q}$ and so $y = (1/x)(xy) \in \mathbb{Q}$, which again would contradict our assumption. Hence $xy \in \mathbb{I}$.

Finally for (iii), suppose $x \neq 0$ and x/y where rational. Then by what we just proved in part (ii), $x = (x/y)y \in \mathbb{I}$ which would be a contradiction. Hence $x/y \in \mathbb{I}$.

Claim 3. *Suppose x and y are irrational. Then $x + y$ and xy may be rational or irrational.*

To see this, let $x = -y = \sqrt{2}$. Then $x \in \mathbb{I}$ and $y \in \mathbb{I}$, but $x + y = 0 \in \mathbb{Q}$

and $xy = 2 \in \mathbb{Q}$. On the other hand, let $x = \sqrt{2}$ and $y = \sqrt{3}$. Then $x \in \mathbb{I}$ and $y \in \mathbb{I}$, but this time $xy = \sqrt{6} \in \mathbb{I}$. The sum $x + y = \sqrt{2} + \sqrt{3}$ is also irrational. To see this, suppose $z = \sqrt{2} + \sqrt{3} \in \mathbb{Q}$. Then by Claim 1 we would have $z^2 = 5 + 2\sqrt{6} \in \mathbb{Q}$. But this would imply $z^2 - 5 = 2\sqrt{6} \in \mathbb{Q}$, which in turn would show $(z^2 - 5)/2 = \sqrt{6} \in \mathbb{Q}$, which is a contradiction since $\sqrt{6}$ is irrational.

Claim 3 shows that sums and products of irrational numbers could be rational or irrational. We can prove a sharper statement:

Claim 4. *Every real number x is the sum of two irrational numbers. If $x \neq 0$, then x is the product of two irrational numbers.*

To see this, first suppose $x \in \mathbb{Q}$. Then, $x - \sqrt{2} \in \mathbb{I}$ (why?) so $x = \sqrt{2} + (x - \sqrt{2})$ is the sum of two irrationals. On the other hand, if $x \neq 0$, then $x/\sqrt{2} \in \mathbb{I}$ (why?) so $x = \sqrt{2}\,(x/\sqrt{2})$ is the product of two irrationals.

Now suppose $x \in \mathbb{I}$. Then $x/2 \in \mathbb{I}$ (why?) so $x = (x/2) + (x/2)$ is the sum of two irrationals. On the other hand, $|x| \in \mathbb{I}$ so $\sqrt{|x|} \in \mathbb{I}$ (why?). Hence $|x| = \sqrt{|x|}\sqrt{|x|}$ is the product of two irrationals. Since x is either $|x|$ or $-|x|$, it easily follows that x itself must be the product of two irrational numbers.