6.23 Integration by partial fractions

We recall that a quotient of two polynomials is called a rational function. Differentiation of a rational function leads to a new rational function which may be obtained by the quotient rule for derivatives. On the other hand, integration of a rational function may lead to functions that are not rational. For example, we have

$$\int \frac{dx}{x} = \log |x| + C \qquad \text{and} \qquad \int \frac{dx}{1+x^2} = \arctan x + C\,.$$

We shall describe a method for computing the integral of any rational function, and we shall find that the result can always be expressed in terms of polynomials, rational functions, inverse tangents, and logarithms.

The basic idea of the method is to decompose a given rational function into a sum of simpler fractions (called partial fractions) that can be integrated by the techniques discussed earlier. We shall describe the general procedure by means of a number of simple examples that illustrate all the essential features of the method.

EXAMPLE 1. In this example we begin with two simple fractions, $1/(x-1)$ and $1/(x+3)$, which we know how to integrate, and see what happens when we form a linear combination of these fractions. For example, if we take twice the first fraction plus three times the second, we obtain

$$\frac{2}{x-1} + \frac{3}{x+3} = \frac{2(x+3)+3(x-1)}{(x-1)(x+3)} = \frac{5x+3}{x^2+2x-3}\,.$$

If, now, we read this formula from right to left, it tells us that the rational function r given by $r(x) = (5x + 3)/(x^2 + 2x - 3)$ has been expressed as a linear combination of $1/(x - 1)$ and $1/(x + 3)$. Therefore, we may evaluate the integral of r by writing

$$\int \frac{5x + 3}{x^2 + 2x - 3}\, dx = 2\int \frac{dx}{x - 1} + 3\int \frac{dx}{x + 3} = 2 \log |x - 1| + 3 \log |x + 3| + C .$$

EXAMPLE 2. The foregoing example suggests a procedure for dealing with integrals of the form $\int (ax + b)/(x^2 + 2x - 3)\, dx$. For example, to evaluate $\int (2x + 5)/(x^2 + 2x - 3)\, dx$, we try to express the integral as a linear combination of $1/(x - 1)$ and $1/(x + 3)$ by writing

$$\frac{2x + 5}{x^2 + 2x - 3} = \frac{A}{x - 1} + \frac{B}{x + 3} \tag{6.55}$$

with constants A and B to be determined. If we can choose A and B so that Equation (6.55) is an identity, then the integral of the fraction on the left is equal to the sum of the integrals of the simpler fractions on the right. To find A and B, we multiply both sides of (6.55) by $(x - 1)(x + 3)$ to remove the fractions. This gives us

$$A(x + 3) + B(x - 1) = 2x + 5 . \tag{6.56}$$

At this stage there are two methods commonly used to find A and B. One method is to equate coefficients of like powers of x in (6.56). This leads to the equations $A + B = 2$ and $3A - B = 5$. Solving this pair of simultaneous equations, we obtain $A = \frac{7}{4}$ and $B = \frac{1}{4}$. The other method involves the substitution of two values of x in (6.56) and leads to another pair of equations for A and B. In this particular case, the presence of the factors $x - 1$ and $x + 3$ suggests that we use the values $x = 1$ and $x = -3$. When we put $x = 1$ in (6.56), the coefficient of B vanishes, and we find $4A = 7$, or $A = \frac{7}{4}$. Similarly, we can make the coefficient of A vanish by putting $x = -3$. This gives us $-4B = -1$, or $B = \frac{1}{4}$. In any event, we have found values of A and B to satisfy (6.55), so we have

$$\int \frac{2x + 5}{x^2 + 2x - 3}\, dx = \frac{7}{4}\int \frac{dx}{x - 1} + \frac{1}{4}\int \frac{dx}{x + 3} = \frac{7}{4} \log |x - 1| + \frac{1}{4} \log |x + 3| + C .$$

It is clear that the method described in Example 2 also applies to integrals of the form $\int f(x)/g(x)\, dx$ in which f is a linear polynomial and g is a quadratic polynomial that can be factored into distinct linear factors with real coefficients, say $g(x) = (x - x_1)(x - x_2)$. In this case the quotient $f(x)/g(x)$ can be expressed as a linear combination of $1/(x - x_1)$ and $1/(x - x_2)$, and integration of $f(x)/g(x)$ leads to a corresponding combination of the logarithmic terms $\log |x - x_1|$ and $\log |x - x_2|$.

The foregoing examples involve rational functions f/g in which the degree of the numerator is less than that of the denominator. A rational function with this property is said to be a *proper* rational function. If f/g is *improper*, that is, if the degree of f is not less than that of g, then we can express f/g as the sum of a polynomial and a proper rational function. In fact, we simply divide f by g to obtain

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)} ,$$

where Q and R are polynomials (called the *quotient* and *remainder*, respectively) such that the remainder has degree less than that of g. For example,

$$\frac{x^3 + 3x}{x^2 - 2x - 3} = x + 2 + \frac{10x + 6}{x^2 - 2x - 3}.$$

Therefore, in the study of integration technique, there is no loss in generality if we restrict ourselves to *proper* rational functions, and from now on we consider $\int f(x)/g(x)\,dx$, where f has degree less than that of g.

A general theorem in algebra states that every proper rational function can be expressed as a finite sum of fractions of the forms

$$\frac{A}{(x + a)^k} \quad \text{and} \quad \frac{Bx + C}{(x^2 + bx + c)^m},$$

where k and m are positive integers and A, B, C, a, b, c are constants with $b^2 - 4c < 0$. The condition $b^2 - 4c < 0$ means that the quadratic polynomial $x^2 + bx + c$ cannot be factored into linear factors with real coefficients or, what amounts to the same thing, the quadratic equation $x^2 + bx + c = 0$ has no real roots. Such a quadratic factor is said to be *irreducible*. When a rational function has been so expressed, we say that it has been decomposed into *partial fractions*. Therefore the problem of integrating this rational function reduces to that of integrating its partial fractions. These may be easily dealt with by the techniques described in the examples which follow.

We shall not bother to prove that partial-fraction decompositions always exist. Instead, we shall show (by means of examples) how to obtain the partial fractions in specific problems. In each case that arises the partial-fraction decomposition can be verified directly.

It is convenient to separate the discussion into cases depending on the way in which the denominator of the quotient $f(x)/g(x)$ can be factored.

CASE 1. The denominator is a product of distinct linear factors. Suppose that $g(x)$ splits into n distinct linear factors, say

$$g(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

Now notice that a linear combination of the form

$$\frac{A_1}{x - x_1} + \cdots + \frac{A_n}{x - x_n}$$

may be expressed as a single fraction with the common denominator $g(x)$, and the numerator of this fraction will be a polynomial of degree $< n$ involving the A's. Therefore, if we can find A's to make this numerator equal to $f(x)$, we shall have the decomposition

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - x_1} + \cdots + \frac{A_n}{x - x_n},$$

and the integral of $f(x)/g(x)$ will be equal to $\sum_{i=1}^{n} A_i \log |x - x_i|$. In the next example, we work out a case with $n = 3$.

EXAMPLE 3. Integrate $\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x}\, dx$.

Solution. Since $x^3 + x^2 - 2x = x(x - 1)(x + 2)$, the denominator is a product of distinct linear factors, and we try to find A_1, A_2, and A_3 such that

$$\frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} = \frac{A_1}{x} + \frac{A_2}{x - 1} + \frac{A_3}{x + 2}.$$

Clearing the fractions, we obtain

$$2x^2 + 5x - 1 = A_1(x - 1)(x + 2) + A_2 x(x + 2) + A_3 x(x - 1).$$

When $x = 0$, we find $-2A_1 = -1$, so $A_1 = \frac{1}{2}$. When $x = 1$, we obtain $3A_2 = 6$, $A_2 = 2$, and when $x = -2$, we find $6A_3 = -3$, or $A_3 = -\frac{1}{2}$. Therefore we have

$$\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x}\, dx = \frac{1}{2}\int \frac{dx}{x} + 2\int \frac{dx}{x - 1} - \frac{1}{2}\int \frac{dx}{x + 2}$$
$$= \tfrac{1}{2} \log |x| + 2 \log |x - 1| - \tfrac{1}{2} \log |x + 2| + C.$$

CASE 2. The denominator is a product of linear factors, some of which are repeated. We illustrate this case with an example.

EXAMPLE 4. Integrate $\int \frac{x^2 + 2x + 3}{(x - 1)(x + 1)^2}\, dx$.

Solution. Here we try to find A_1, A_2, A_3 so that

(6.57) $$\frac{x^2 + 2x + 3}{(x - 1)(x + 1)^2} = \frac{A_1}{x - 1} + \frac{A_2}{x + 1} + \frac{A_3}{(x + 1)^2}.$$

We need both $A_2/(x + 1)$ and $A_3/(x + 1)^2$ as well as $A_1/(x - 1)$ in order to get a polynomial of degree two in the numerator and to have as many constants as equations when we try to determine the A's. Clearing the fractions, we obtain

(6.58) $$x^2 + 2x + 3 = A_1(x + 1)^2 + A_2(x - 1)(x + 1) + A_3(x - 1).$$

Substituting $x = 1$, we find $4A_1 = 6$, so $A_1 = \frac{3}{2}$. When $x = -1$, we obtain $-2A_3 = 2$ and $A_3 = -1$. We need one more equation to determine A_2. Since there are no other choices of x that will make any factor vanish, we choose a convenient x that will help to simplify the calculations. For example, the choice $x = 0$ leads to the equation $3 = A_1 - A_2 - A_3$ from which we find $A_2 = -\frac{1}{2}$. An alternative method is to differentiate both sides of (6.58) and then substitute a convenient x. Differentiation of (6.58) leads to the equation

$$2x + 2 = 2A_1(x + 1) + A_2(x - 1) + A_2(x + 1) + A_3\,,$$

and, if we put $x = -1$, we find $0 = -2A_2 + A_3$, so $A_2 = \frac{1}{2}A_3 = -\frac{1}{2}$, as before. Therefore we have found A's to satisfy (6.57), so we have

$$\int \frac{x^2 + 2x + 3}{(x - 1)(x + 1)^2}\,dx = \frac{3}{2}\int \frac{dx}{x - 1} - \frac{1}{2}\int \frac{dx}{x + 1} - \int \frac{dx}{(x + 1)^2}$$

$$= \frac{3}{2}\log|x - 1| - \frac{1}{2}\log|x + 1| + \frac{1}{x + 1} + C\,.$$

If, on the left of (6.57), the factor $(x + 1)^3$ had appeared instead of $(x + 1)^2$, we would have added an extra term $A_4/(x + 1)^3$ on the right. More generally, if a linear factor $x + a$ appears p times in the denominator, then for this factor we must allow for a sum of p terms, namely

$$\sum_{k=1}^{p} \frac{A_k}{(x + a)^k}\,, \tag{6.59}$$

where the A's are constants. A sum of this type is to be used for each repeated linear factor.

CASE 3. The denominator contains irreducible quadratic factors, none of which are repeated.

EXAMPLE 5. Integrate $\displaystyle\int \frac{3x^2 + 2x - 2}{x^3 - 1}\,dx$.

Solution. The denominator can be split as the product $x^3 - 1 = (x - 1)(x^2 + x + 1)$, where $x^2 + x + 1$ is irreducible, and we try a decomposition of the form

$$\frac{3x^2 + 2x - 2}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}\,.$$

In the fraction with denominator $x^2 + x + 1$, we have used a linear polynomial $Bx + C$ in the numerator in order to have as many constants as equations when we solve for A, B, C. Clearing the fractions and solving for A, B, and C, we find $A = 1$, $B = 2$, and $C = 3$. Therefore we have

$$\int \frac{3x^2 + 2x - 2}{x^3 - 1}\,dx = \int \frac{dx}{x - 1} + \int \frac{2x + 3}{x^2 + x + 1}\,dx\,.$$

The first integral on the right is $\log|x - 1|$. To evaluate the second integral, we write

$$\int \frac{2x + 3}{x^2 + x + 1}\,dx = \int \frac{2x + 1}{x^2 + x + 1}\,dx + \int \frac{2}{x^2 + x + 1}\,dx$$

$$= \log(x^2 + x + 1) + 2\int \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}}\,.$$

If we let $u = x + \frac{1}{2}$ and $\alpha = \sqrt{\frac{3}{4}}$, the last integral is

$$2\int \frac{du}{u^2 + \alpha^2} = \frac{2}{\alpha} \arctan \frac{u}{\alpha} = \frac{4}{3}\sqrt{3} \arctan \frac{2x + 1}{\sqrt{3}} .$$

Therefore, we have

$$\int \frac{3x^2 + 2x - 2}{x^3 - 1}\, dx = \log|x - 1| + \log(x^2 + x + 1) + \frac{4}{3}\sqrt{3} \arctan \frac{2x + 1}{\sqrt{3}} + C .$$

CASE 4. The denominator contains irreducible quadratic factors, some of which are repeated. Here the situation is analogous to Case 2. In the partial-fraction decomposition of $f(x)/g(x)$ we allow, first of all, a sum of the form (6.59) for each linear factor, as already described. In addition, if an irreducible quadratic factor $x^2 + bx + c$ is repeated m times, we allow a sum of m terms, namely

$$\sum_{k=1}^{m} \frac{B_k x + C_k}{(x^2 + bx + c)^k} ,$$

where each numerator is linear.

EXAMPLE 6. Integrate $\displaystyle\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x - 1)(x^2 + 2)^2}\, dx$.

Solution. We write

$$\frac{x^4 - x^3 + 2x^2 - x + 2}{(x - 1)(x^2 + 2)^2} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2} .$$

Clearing the fractions and solving for A, B, C, D, and E, we find that

$$A = \tfrac{1}{3}, \qquad B = \tfrac{2}{3}, \qquad C = -\tfrac{1}{3}, \qquad D = -1, \qquad E = 0 .$$

Therefore, we have

$$\begin{aligned}\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x - 1)(x^2 + 2)^2}\, dx &= \frac{1}{3}\int \frac{dx}{x - 1} + \int \frac{\frac{2}{3}x - \frac{1}{3}}{x^2 + 2}\, dx - \int \frac{x\, dx}{(x^2 + 2)^2} \\ &= \frac{1}{3}\int \frac{dx}{x - 1} + \frac{1}{3}\int \frac{2x\, dx}{x^2 + 2} - \frac{1}{3}\int \frac{dx}{x^2 + 2} - \frac{1}{2}\int \frac{2x\, dx}{(x^2 + 2)^2} \\ &= \frac{1}{3}\log|x - 1| + \frac{1}{3}\log(x^2 + 2) - \frac{\sqrt{2}}{6} \arctan \frac{x}{\sqrt{2}} \\ &\quad + \frac{1}{2}\frac{1}{x^2 + 2} + C .\end{aligned}$$

The foregoing examples are typical of what happens in general. The problem of integrating a proper rational function reduces to that of calculating integrals of the forms

$$\int \frac{dx}{(x+a)^n}, \qquad \int \frac{x\,dx}{(x^2+bx+c)^m}, \qquad \text{and} \qquad \int \frac{dx}{(x^2+bx+c)^m}.$$

The first integral is $\log|x+a|$ if $n = 1$ and $(x+a)^{1-n}/(1-n)$ if $n > 1$. To treat the other two, we express the quadratic as a sum of two squares by writing

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) = u^2 + \alpha^2,$$

where $u = x + b/2$ and $\alpha = \frac{1}{2}\sqrt{4c - b^2}$. (This is possible because $4c - b^2 > 0$.) The substitution $u = x + b/2$ reduces the problem to that of computing

$$\int \frac{u\,du}{(u^2+\alpha^2)^m} \qquad \text{and} \qquad \int \frac{du}{(u^2+\alpha^2)^m}. \tag{6.60}$$

The first of these is $\frac{1}{2}\log(u^2+\alpha^2)$ if $m = 1$, and $\frac{1}{2}(u^2+\alpha^2)^{1-m}/(1-m)$ if $m > 1$. When $m = 1$, the second integral in (6.60) is evaluated by the formula

$$\int \frac{du}{u^2+\alpha^2} = \frac{1}{\alpha}\arctan\frac{u}{\alpha} + C.$$

The case $m > 1$ may be reduced to the case $m = 1$ by repeated application of the recursion formula

$$\int \frac{du}{(u^2+\alpha^2)^m} = \frac{1}{2\alpha^2(m-1)}\frac{u}{(u^2+\alpha^2)^{m-1}} + \frac{2m-3}{2\alpha^2(m-1)}\int \frac{du}{(u^2+\alpha^2)^{m-1}},$$

which is obtained by integration by parts. This discussion shows that every rational function may be integrated in terms of polynomials, rational functions, inverse tangents, and logarithms.