7.9 Further remarks on the error in Taylor's formula. The o-notation

If f has a continuous $(n+1)$st derivative in some interval containing a point a, we may write Taylor's formula in the form

$$(7.17) \qquad f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + E_n(x).$$

Suppose we restrict x to lie in some closed interval $[a-c, a+c]$ about a, in which $f^{(n+1)}$ is continuous. Then $f^{(n+1)}$ is bounded on this interval and hence satisfies an inequality of the form

$$|f^{(n+1)}(t)| \le M,$$

where $M > 0$. Hence, by Theorem 7.7, we have the error estimate

$$|E_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}$$

for each x in $[a-c, a+c]$. If we keep $x \ne a$ and divide this inequality by $|x-a|^n$, we find that

$$0 \le \left| \frac{E_n(x)}{(x-a)^n} \right| \le \frac{M}{(n+1)!}|x-a|.$$

If now we let $x \to a$, we see that $E_n(x)/(x-a)^n \to 0$. We describe this by saying that the error $E_n(x)$ is of smaller order than $(x-a)^n$ as $x \to a$.

In other words, under the conditions stated, $f(x)$ may be approximated near a by a polynomial in $(x-a)$ of degree n, and the error in this approximation is of smaller order than $(x-a)^n$ as $x \to a$.

A special notation, introduced in 1909 by E. Landau,† is particularly appropriate when used in connection with Taylor's formula. This is called the o-notation (the little-oh notation) and it is defined as follows.

DEFINITION. *Assume $g(x) \ne 0$ for all $x \ne a$ in some interval containing a. The notation*

$$f(x) = o(g(x)) \qquad \text{as} \quad x \to a$$

means that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

The symbol $f(x) = o(g(x))$ is read "$f(x)$ is little-oh of $g(x)$," or "$f(x)$ is of smaller order than $g(x)$," and it is intended to convey the idea that for x near a, $f(x)$ is small compared with $g(x)$.

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† Edmund Landau (1877–1938) was a famous German mathematician who made many important contributions to mathematics. He is best known for his lucid books in analysis and in the theory of numbers.

EXAMPLE 1. $f(x) = o(1)$ as $x \to a$ means that $f(x) \to 0$ as $x \to a$.

EXAMPLE 2. $f(x) = o(x)$ as $x \to 0$ means that $\dfrac{f(x)}{x} \to 0$ as $x \to 0$.

An equation of the form $f(x) = h(x) + o(g(x))$ is understood to mean that $f(x) - h(x) = o(g(x))$ or, in other words, $[f(x) - h(x)]/g(x) \to 0$ as $x \to a$.

EXAMPLE 3. We have $\sin x = x + o(x)$ because $\dfrac{\sin x - x}{x} = \dfrac{\sin x}{x} - 1 \to 0$ as $x \to 0$.

The foregoing remarks concerning the error in Taylor's formula can now be expressed in the o-notation. We may write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o((x-a)^n) \qquad \text{as} \quad x \to a\,,$$

whenever the derivative $f^{(n+1)}$ is continuous in some closed interval containing the point a. This expresses, in a brief way, the fact that the error term is small compared to $(x - a)^n$ when x is near a. In particular, from the discussion of earlier sections, we have the following examples of Taylor's formula expressed in the o-notation:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n) \qquad \text{as} \quad x \to 0\,.$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n) \qquad \text{as} \quad x \to 0\,.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n) \qquad \text{as} \quad x \to 0\,.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + o(x^{2n}) \qquad \text{as} \quad x \to 0\,.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1}) \qquad \text{as} \quad x \to 0\,.$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + o(x^{2n}) \qquad \text{as} \quad x \to 0\,.$$

In calculations involving Taylor approximations, it often becomes necessary to combine several terms involving the o-symbol. A few simple rules for manipulating o-symbols are discussed in the next theorem. These cover most situations that arise in practice.

THEOREM 7.8. ALGEBRA OF o-SYMBOLS. *As $x \to a$, we have the following:*

(a) $o(g(x)) \pm o(g(x)) = o(g(x))$.
(b) $o(cg(x)) = o(g(x))$ *if* $c \neq 0$.
(c) $f(x) \cdot o(g(x)) = o(f(x)g(x))$.
(d) $o(o(g(x))) = o(g(x))$.

(e) $\dfrac{1}{1 + g(x)} = 1 - g(x) + o(g(x))$ *if* $g(x) \to 0$ *as* $x \to a$.

Proof. The statement in part (a) is understood to mean that if $f_1(x) = o(g(x))$ and if $f_2(x) = o(g(x))$, then $f_1(x) \pm f_2(x) = o(g(x))$. But since we have

$$\frac{f_1(x) \pm f_2(x)}{g(x)} = \frac{f_1(x)}{g(x)} \pm \frac{f_2(x)}{g(x)},$$

each term on the right tends to 0 as $x \to a$, so part (a) is proved. The statements in (b), (c), and (d) are proved in a similar way.

To prove (e), we use the algebraic identity

$$\frac{1}{1+u} = 1 - u + u\frac{u}{1+u}$$

with u replaced by $g(x)$ and then note that $\dfrac{g(x)}{1 + g(x)} \to 0$ as $x \to a$.

EXAMPLE 1. Prove that $\tan x = x + \frac{1}{3}x^3 + o(x^3)$ as $x \to 0$.

Solution. We use the Taylor approximations for the sine and cosine. From part (e) of Theorem 7.8, with $g(x) = -\frac{1}{2}x^2 + o(x^3)$, we have

$$\frac{1}{\cos x} = \frac{1}{1 - \frac{1}{2}x^2 + o(x^3)} = 1 + \frac{1}{2}x^2 + o(x^2) \quad \text{as} \quad x \to 0 .$$

Therefore, we have

$$\tan x = \frac{\sin x}{\cos x} = \left(x - \frac{1}{6}x^3 + o(x^4)\right)\left(1 + \frac{1}{2}x^2 + o(x^2)\right) = x + \frac{1}{3}x^3 + o(x^3) .$$

EXAMPLE 2. Prove that $(1 + x)^{1/x} = e \cdot \left(1 - \dfrac{x}{2} + \dfrac{11x^2}{24} + o(x^2)\right)$ as $x \to 0$.

Solution. Since $(1 + x)^{1/x} = e^{(1/x)\log(1+x)}$, we begin with a polynomial approximation to $\log(1 + x)$. Taking a cubic approximation, we have

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3), \qquad \frac{\log(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} + o(x^2),$$

and so we obtain

$$(7.18) \qquad (1+x)^{1/x} = \exp\,(1 - x/2 + x^2/3 + o(x^2)) = e \cdot e^u,$$

where $u = -x/2 + x^2/3 + o(x^2)$. But as $u \to 0$, we have $e^u = 1 + u + \frac{1}{2}u^2 + o(u^2)$, so we obtain

$$e^u = 1 - \frac{x}{2} + \frac{x^2}{3} + o(x^2) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} + o(x^2)\right)^2 + o(x^2) = 1 - \frac{x}{2} + \frac{11x^2}{24} + o(x^2)\,.$$

When we use this in Equation (7.18), we obtain the desired formula.

7.10 Applications to indeterminate forms

We have already illustrated how polynomial approximations are used in the computation of function values. They can also be used as an aid in the calculation of limits. We illustrate with some examples.

EXAMPLE 1. If a and b are positive numbers, determine the limit

$$\lim_{x \to 0} \frac{a^x - b^x}{x}\,.$$

Solution. We cannot solve this problem by computing the limit of the numerator and denominator separately, because the denominator tends to 0 and the quotient theorem on limits is not applicable. The numerator in this case also tends to 0 and the quotient is said to assume the "*indeterminate form* 0/0" as $x \to 0$. Taylor's formula and the o-notation often enable us to calculate the limit of an indeterminate form like this one very simply. The idea is to approximate the numerator $a^x - b^x$ by a polynomial in x, then divide by x and let $x \to 0$. We could apply Taylor's formula directly to $f(x) = a^x - b^x$ but, since $a^x = e^{x \log a}$ and $b^x = e^{x \log b}$, it is simpler in this case to use the polynomial approximations already derived for the exponential function. If we begin with the linear approximation

$$e^t = 1 + t + o(t) \qquad \text{as} \quad t \to 0$$

and replace t by $x \log a$ and $x \log b$, respectively, we find

$$a^x = 1 + x \log a + o(x) \qquad \text{and} \qquad b^x = 1 + x \log b + o(x) \qquad \text{as} \quad x \to 0\,.$$

Here we have used the fact that $o(x \log a) = o(x)$ and $o(x \log b) = o(x)$. If now we subtract and note that $o(x) - o(x) = o(x)$, we find $a^x - b^x = x(\log a - \log b) + o(x)$. Dividing by x and using the relation $o(x)/x = o(1)$, we obtain

$$\frac{a^x - b^x}{x} = \log\frac{a}{b} + o(1) \to \log\frac{a}{b} \qquad \text{as} \quad x \to 0\,.$$

EXAMPLE 2. Prove that $\lim_{x\to 0} \frac{1}{x}\left(\cot x - \frac{1}{x}\right) = -\frac{1}{3}$.

Solution. We use Example 1 of Section 7.9, and Theorem 7.8(e) to write

$$\cot x = \frac{1}{\tan x} = \frac{1}{x + \frac{1}{3}x^3 + o(x^3)} = \frac{1}{x}\,\frac{1}{1 + \frac{1}{3}x^2 + o(x^2)}$$
$$= \frac{1}{x}\left(1 - \frac{1}{3}x^2 + o(x^2)\right) = \frac{1}{x} - \frac{1}{3}x + o(x)\,.$$

Hence, we have

$$\frac{1}{x}\left(\cot x - \frac{1}{x}\right) = -\frac{1}{3} + o(1) \to -\frac{1}{3} \quad \text{as} \quad x \to 0\,.$$

EXAMPLE 3. Prove that $\lim_{x\to 0} \frac{\log(1 + ax)}{x} = a$ for every real a.

Solution. If $a = 0$, the result holds trivially. If $a \neq 0$, we use the linear approximation $\log(1 + x) = x + o(x)$. Replacing x by ax, we obtain $\log(1 + ax) = ax + o(ax) = ax + o(x)$. Dividing by x and letting $x \to 0$, we obtain the limit a.

EXAMPLE 4. Prove that for every real a, we have

$$\lim_{x\to 0}(1 + ax)^{1/x} = e^a\,. \tag{7.19}$$

Solution. We simply note that $(1 + ax)^{1/x} = e^{(1/x)\log(1+ax)}$ and use the result of Example 3 along with the continuity of the exponential function.

Replacing ax by y in (7.19), we find another important limit relation:

$$\lim_{y\to 0}(1 + y)^{a/y} = e^a\,.$$

Sometimes these limit relations are taken as the starting point for the theory of the exponential function.

7.11 Exercises

1. Find a quadratic polynomial $P(x)$ such that $2^x = P(x) + o(x^2)$ as $x \to 0$.
2. Find a cubic polynomial $P(x)$ such that $x \cos x = P(x) + o((x - 1)^3)$ as $x \to 1$.
3. Find the polynomial $P(x)$ of smallest degree such that $\sin(x - x^2) = P(x) + o(x^6)$ as $x \to 0$.
4. Find constants a, b, c such that $\log x = a + b(x - 1) + c(x - 1)^2 + o((x - 1)^2)$ as $x \to 1$.
5. Recall that $\cos x = 1 - \frac{1}{2}x^2 + o(x^3)$ as $x \to 0$. Use this to prove that $x^{-2}(1 - \cos x) \to \frac{1}{2}$ as $x \to 0$. In a similar way, find the limit of $x^{-4}(1 - \cos 2x - 2x^2)$ as $x \to 0$.

Evaluate the limits in Exercises 6 through 29.

6. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$.

7. $\lim_{x\to 0} \dfrac{\tan 2x}{\sin 3x}$.

8. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$.

9. $\lim_{x\to 0} \dfrac{\log(1+x)}{e^{2x}-1}$.

10. $\lim_{x\to 0} \dfrac{1-\cos^2 x}{x\tan x}$.

11. $\lim_{x\to 0} \dfrac{\sin x}{\arctan x}$.

12. $\lim_{x\to 0} \dfrac{a^x - 1}{b^x - 1}$, $\quad b \neq 1$.

13. $\lim_{x\to 1} \dfrac{\log x}{x^2 + x - 2}$.

14. $\lim_{x\to 0} \dfrac{1-\cos x^2}{x^2 \sin x^2}$.

15. $\lim_{x\to 0} \dfrac{x(e^x+1) - 2(e^x-1)}{x^3}$.

16. $\lim_{x\to 0} \dfrac{\log(1+x) - x}{1-\cos x}$.

17. $\lim_{x\to \frac{1}{2}\pi} \dfrac{\cos x}{x - \frac{1}{2}\pi}$.

18. $\lim_{x\to 1} \dfrac{[\sin(\pi/2x)](\log x)}{(x^3+5)(x-1)}$.

19. $\lim_{x\to 0} \dfrac{\cosh x - \cos x}{x^2}$.

20. $\lim_{x\to 0} \dfrac{3\tan 4x - 12\tan x}{3\sin 4x - 12\sin x}$.

21. $\lim_{x\to 0} \dfrac{a^x - a^{\sin x}}{x^3}$.

22. $\lim_{x\to 0} \dfrac{\cos(\sin x) - \cos x}{x^4}$.

23. $\lim_{x\to 1} x^{1/(1-x)}$.

24. $\lim_{x\to 0} (x + e^{2x})^{1/x}$.

25. $\lim_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x}$.

26. $\lim_{x\to 0} \left(\dfrac{(1+x)^{1/x}}{e}\right)^{1/x}$.

27. $\lim_{x\to 0} \left(\dfrac{\arcsin x}{x}\right)^{1/x^2}$.

28. $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$.

29. $\lim_{x\to 1} \left(\dfrac{1}{\log x} - \dfrac{1}{x-1}\right)$.

30. For what value of the constant a will $x^{-2}(e^{ax} - e^x - x)$ tend to a finite limit as $x \to 0$? What is the value of this limit?

31. Given two functions f and g with derivatives in some interval containing 0, where g is positive. Assume also $f(x) = o(g(x))$ as $x \to 0$. Prove or disprove each of the following statements:

(a) $\int_0^x f(t)\,dt = o\left(\int_0^x g(t)\,dt\right)$ as $x \to 0$, $\qquad$ (b) $f'(x) = o(g'(x))$ as $x \to 0$.

32. (a) If $g(x) = o(1)$ as $x \to 0$, prove that

$$\frac{1}{1+g(x)} = 1 - g(x) + g^2(x) + o(g^2(x)) \quad \text{as} \quad x \to 0.$$

(b) Use part (a) to prove that $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + o(x^5) \quad$ as $\quad x \to 0$.

33. A function f has a continuous third derivative everywhere and satisfies the relation

$$\lim_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{1/x} = e^3.$$