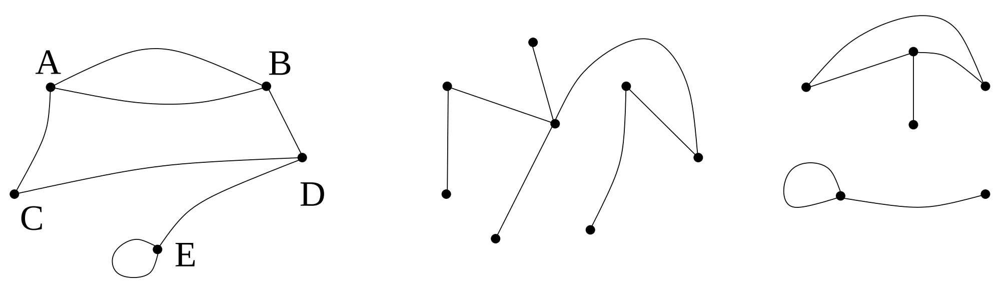
MAT 160, PROBLEM SEMINAR, WEEK OF 3/1/99

PROBLEM SET 6: GRAPHS

For the following problems, you need to know these definitions:
• A *graph* consists of a finite set of points called *vertices* and a finite number of arcs called *edges* joining some of the vertices. Examples:



(a) (b) (c)

• The number of edges attached to each vertex is called the *degree* of that vertex. So for example, in the above graph (a), vertices A, B, D, E have degree 3, while vertex C has degree 2.
• A graph is called *connected* if you can connect any two vertices by a sequence of edges. Thus graphs (a) and (b) are connected, while (c) is not connected.
• A graph is called a *tree* if it contains no "loops" in it. Equivalently, if there is at most one way to connect any two vertices by edges. For example, graph (b) above is a tree, but (a) and (c) are not, since both of them have non-trivial loops in them (in (a), you can find many loops like A→B→D→C→A, or A→B→A or E→E).

Problem 36. In any graph, show that the sum of the degrees of all the vertices is twice the number of edges. (In particular, it follows that this sum must be an even number.)

Problem 37. In a town with only 25 telephones, is it possible to connect each telephone by wires to exactly 5 other telephones? (*Hint*: Consider the graph in which vertices represent the telephones and edges represent wires. Use problem 36.)

Problem 38. There are 30 students in a class. Can it happen that 9 of them have 3 friends each (in the class), 11 of them have 4 friends each, and 10 of them have 5 friends each? (*Hint*: Again, consider the graph in which vertices represent the students and connect any two friends by an edge.)

Problem 39. In Mathland there are nine cities called $1, 2, \cdots, 9$ (they give cities numbers rather than names!). There is a direct flight from one city to another if and only if the two-digit number formed by the first city then the second city is divisible by 3. For example, one can fly from city 2 to city 4 (since 24 is divisible by 3) but there is no flight from city 7 to city 1 (since 71 is not divisible by 3). Is there any way for a passenger to get from city 1 to city 9?

Problem 40. Can you draw 9 straight line segments in the plane, each of which intersecting exactly 3 others? (*Hint*: Assume there is such a configuration. Consider the graph in which vertices represent the line segments and connect two vertices by an edge if and only if the corresponding line segments intersect.)

Problem 41. In NPBM (National Park for Bored Mathematicians) there is a lake with 7 islands. There are 1, 3, or 5 bridges leading to each island. Is it true that at least one of these bridges must lead to the shore of the lake? (*Hint*: Assume all the bridges are among the islands and hence none of them leads to the shore. Consider the graph in which vertices represent the islands and edges reresent the bridges connecting them.)

Problem 42. Suppose that you have a connected tree T.

(a) Show that T must have at least one vertex with degree 1, i.e., with only 1 edge attached to it (in the example (b) above, all three bottom vertices have this property, as well as the middle vertex in the top). Such a vertex is called a *root* of the tree.

(b) Show that the number of vertices of T is one plus the number of edges of T. (*Hint*: Use induction on the number of vertices of T. For a tree with only $n = 1$ vertex, the result is trivial. Assume the result is true for all connected trees with n vertices. Take any connected tree with $n+1$ vertices and find a root by part (a) above. After removing this root with the edge attached to it, you get a connected tree of n vertices. Apply the induction hypothesis now.)