Math 201 First Midterm Solutions

Problem 1. [8 points] Let

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{b} = 2\mathbf{i} + 3\mathbf{j}.$$

Find the parametric equation of the line passing through the point $(5, -1, 4)$ in the direction of the vector $\mathbf{a} \times \mathbf{b}$.

Solution. We compute

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}.$$

Thus, the line passes through the tip of $\mathbf{r}_0 = \langle 5, -1, 4 \rangle$ and is in the direction of the vector $\mathbf{v} = \langle -3, 2, 7 \rangle$. So it has the equation

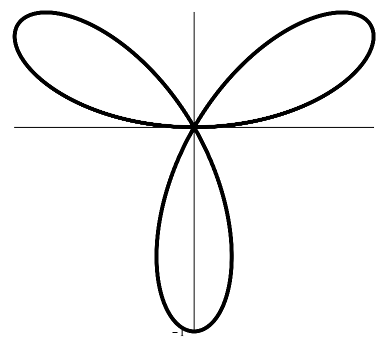
$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = \langle 5, -1, 4 \rangle + t \langle -3, 2, 7 \rangle = \langle 5 - 3t, -1 + 2t, 4 + 7t \rangle,$$

which can be written in parametric form

$$\begin{cases} x = 5 - 3t \\ y = -1 + 2t \\ z = 4 + 7t \end{cases} \qquad (-\infty < t < +\infty).$$

Problem 2. [10 points] Find the area of the region bounded by the "propeller" with the polar equation

$$r = \sin(3\theta).$$



Solution. By symmetry, it suffices to find the area of the right-hand "blade" and multiply it by 3. The part of the curve corresponding to this blade starts at $\theta = 0$ and ends at the next zero of the function $\sin(3\theta)$, namely at $\theta = \pi/3$. Hence

$$\begin{aligned} \text{Area of the right-hand blade} &= \tfrac{1}{2} \int_0^{\pi/3} r^2 \, d\theta \\ &= \tfrac{1}{2} \int_0^{\pi/3} \sin^2(3\theta) \, d\theta \\ &= \tfrac{1}{4} \int_0^{\pi/3} (1 - \cos(6\theta)) \, d\theta \qquad \left(\text{using } \sin^2 x = \tfrac{1 - \cos(2x)}{2}\right) \\ &= \tfrac{1}{4} \left[\theta - \tfrac{1}{6} \sin(6\theta) \right] \Big|_0^{\pi/3} \\ &= \tfrac{1}{4} \left[\left(\tfrac{\pi}{3} - \tfrac{1}{6} \sin(2\pi) \right) - \left(0 - \tfrac{1}{6} \sin(0) \right) \right] = \tfrac{\pi}{12}. \end{aligned}$$

Hence the total area of the propeller is $3 \cdot \pi/12 = \pi/4$.

Problem 3. [4+8 points] Consider the curve C defined by the vector function

$$\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}.$$

(i) At what point on C does the tangent vector have the same direction as $3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$?

Solution. The tangent vector to C at time t is given by the derivative

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j} + \frac{1}{t}\mathbf{k}.$$

This vector has the same direction as $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ if and only if $\mathbf{r}'(t) = c\mathbf{v}$ for some scalar c. We have

$$\mathbf{r}'(t) = c\mathbf{v} \iff 2t\mathbf{i} + 2\mathbf{j} + \frac{1}{t}\mathbf{k} = 3c\mathbf{i} + 6c\mathbf{j} + 6c\mathbf{k} \iff \begin{cases} 2t = 3c \\ 2 = 6c \\ \frac{1}{t} = 6c. \end{cases}$$

From the second equation we have $c = 1/3$. Plugging this into the first or third equation, we obtain $t = 1/2$. Since $\mathbf{r}(1/2) = (1/4)\mathbf{i} + \mathbf{j} - \ln 2\,\mathbf{k}$, it follows that the tangent vector to C at the point $(1/4, 1, -\ln 2)$ has the same direction as $\mathbf{v}$.

(ii) Find the arc-length of the part of C from the point $(1, 2, 0)$ to the point $(e^2, 2e, 1)$.

Solution. Looking at the formula for $\mathbf{r}(t)$, we see that the points $(1, 2, 0)$ and $(e^2, 2e, 1)$ on C correspond to the times $t = 1$ and $t = e$. We have for $t > 0$,

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\left(2t + \frac{1}{t}\right)^2} = 2t + \frac{1}{t}.$$

Hence, by the arc-length formula,

$$\begin{aligned} \text{Arc-length from } (1,2,0) \text{ to } (e^2, 2e, 1) &= \int_1^e |\mathbf{r}'(t)|\,dt \\ &= \int_1^e (2t + \frac{1}{t})\,dt \\ &= \left[t^2 + \ln t\right]\Big|_1^e \\ &= \left[(e^2 + \underbrace{\ln e}_{1}) - (1 + \underbrace{\ln 1}_{0})\right] = e^2 \end{aligned}$$

Problem 4. [7+3 points] Consider the helix defined by the vector function

$$\mathbf{r}(t) = R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j} + ct\,\mathbf{k}.$$

(i) Compute the curvature κ of this helix. Verify that the answer depends on R and c but is independent of t.

Solution. By differentiation, we have

$$\mathbf{r}'(t) = -R\sin t\,\mathbf{i} + R\cos t\,\mathbf{j} + c\,\mathbf{k}$$
$$\mathbf{r}''(t) = -R\cos t\,\mathbf{i} - R\sin t\,\mathbf{j}.$$

Hence,

$$\begin{aligned}\mathbf{r}'(t)\times\mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin t & R\cos t & c \\ -R\cos t & -R\sin t & 0 \end{vmatrix} \\ &= cR\sin t\,\mathbf{i} - cR\cos t\,\mathbf{j} + (R^2\sin^2 t + R^2\cos^2 t)\,\mathbf{k} \\ &= cR\sin t\,\mathbf{i} - cR\cos t\,\mathbf{j} + R^2\,\mathbf{k}.\end{aligned}$$

This gives

$$\begin{aligned}|\mathbf{r}'(t)\times\mathbf{r}''(t)| &= \sqrt{c^2R^2\sin^2 t + c^2R^2\cos^2 t + R^4} \\ &= \sqrt{c^2R^2 + R^4} \\ &= \sqrt{R^2(c^2+R^2)} \\ &= R\sqrt{R^2+c^2}.\end{aligned}$$

On the other hand,

$$|\mathbf{r}'(t)| = \sqrt{R^2\sin^2 t + R^2\cos^2 t + c^2} = \sqrt{R^2+c^2}.$$

We now apply the curvature formula to find κ:

$$\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{R\sqrt{R^2+c^2}}{(\sqrt{R^2+c^2})^3} = \frac{R}{R^2+c^2}.$$

(ii) What happens to κ as $c \to 0$ or $c \to +\infty$? Would you have expected these answers before any computation?

Solution. By the formula we just found for κ,

$$\lim_{c\to 0}\kappa = \lim_{c\to 0}\frac{R}{R^2+c^2} = \frac{1}{R}.$$

This was expected, because as $c \to 0$ the helix tends to the circle of radius R in the xy-plane, whose curvature, as we know, is $1/R$. Similarly,

$$\lim_{c\to +\infty}\kappa = \lim_{c\to +\infty}\frac{R}{R^2+c^2} = 0.$$

Again, this was expected, because as $c \to +\infty$ the helix tends to the vertical line passing through $(1,0,0)$, whose curvature is 0.