Math 201 Second Midterm Solutions

Problem 1. [6+4 points] Suppose $w = f(x, y, z)$ is a differentiable function of three variables x, y, z and

$$x = r^2 + s^2, \qquad y = \cos(r+s), \qquad z = \sin(r+s).$$

We may therefore consider w as a function of r, s.

(i) Use the chain rule to express the partial derivatives w_r and w_s in terms of the partial derivatives f_x, f_y, f_z.

Solution. By the chain rule,

$$w_r = f_x \cdot x_r + f_y \cdot y_r + f_z \cdot z_r = f_x \cdot 2r - f_y \cdot \sin(r+s) + f_z \cdot \cos(r+s)$$
$$w_s = f_x \cdot x_s + f_y \cdot y_s + f_z \cdot z_s = f_x \cdot 2s - f_y \cdot \sin(r+s) + f_z \cdot \cos(r+s).$$

(ii) Suppose $\nabla f(2,1,0) = \langle 5, -3, 7 \rangle$. Compute the partial derivatives w_r and w_s when $r = 1$ and $s = -1$.

Solution. Note that when $r = 1$ and $s = -1$, we have $x = 2, y = 1, z = 0$. Since

$$\nabla f(2,1,0) = \langle f_x, f_y, f_z \rangle = \langle 5, -3, 7 \rangle,$$

we obtain

$$f_x = 5, f_y = -3, f_z = 7$$

at the point $(x, y, z) = (2, 1, 0)$. Substituting these values along with $r = 1, s = -1$ into the formulas for w_r, w_s that we found in part (i), we obtain

$$w_r = (5)(2) - (-3)\sin(0) + (7)\cos(0) = 17$$
$$w_s = (5)(-2) - (-3)\sin(0) + (7)\cos(0) = -3.$$

Problem 2. [8 points] Use differentials to estimate the error in measuring the volume V of a cylindrical tank of base radius $r = 5$ ft and height $h = 25$ ft, if r and h have been measured with an error of no more than 0.01 ft. Is V more sensitive to the error in measuring r or h?

Solution. The volume V is a cylinder of base radius r and height h is given by

$$V = \pi r^2 h.$$

Taking the differential gives

$$dV = V_r\, dr + V_h\, dh = 2\pi r h\, dr + \pi r^2\, dh.$$

Thus, when $r = 5, h = 25$, we have

$$dV = 250\pi\, dr + 25\pi\, dh. \qquad (*)$$

Since by the assumption $|dr|$ and $|dh|$ are at most 0.01, we obtain the estimate

$$|dV| \leq 250\pi \cdot 0.01 + 25\pi \cdot 0.01 \approx 8.64.$$

Thus, we expect an error of no more than 8.64 ft^3 in measuring the volume. (This may seem like a large error, but it's actually quite reasonable: Since the measurement gives $V = (\pi)(5^2)(25) \approx 1963.50$ ft^3, an error of 8.64 ft^3 is a mere $8.64/1963.50 \approx 0.44\%$.)

As to which measurement the volume is more sensitive to, look at $(*)$: A small error in r would magnify by a factor of 250π in V, while a small error in h would magnify only by a factor of 25π. So for the given parameters, V is more sensitive to the measurement of the base radius r.

Problem 3. [6+4 points] Let $f(x,y,z) = 2\ln(x+y) + 3xz^2$.

(i) Find the directional derivative of f at the point $(1,0,1)$ in the direction of the vector $\mathbf{v} = \langle 3,-1,2\rangle$.

Solution. We first compute the gradient vector at $(1,0,1)$:

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle \frac{2}{x+y} + 3z^2, \frac{2}{x+y}, 6xz\rangle,$$

so

$$\nabla f(1,0,1) = \langle 5,2,6\rangle.$$

The unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{14}}\langle 3,-1,2\rangle.$$

Hence,

$$\begin{aligned} D_{\mathbf{u}} f(1,0,1) &= \nabla f(1,0,1) \cdot \mathbf{u} \\ &= \langle 5,2,6\rangle \cdot \frac{1}{\sqrt{14}}\langle 3,-1,2\rangle \\ &= \frac{1}{\sqrt{14}}((5)(3) + (2)(-1) + (6)(2)) \\ &= \frac{25}{\sqrt{14}}. \end{aligned}$$

(ii) Write the equation of the tangent plane to the level surface $f(x,y,z) = 3$ at the point $(1,0,1)$.

Solution. The gradient of f is orthogonal to the level surface $f = 3$ at $(1,0,1)$. Hence the vector $\nabla f(1,0,1) = \langle 5,2,6\rangle$ serves as the normal to the tangent plane at this point. Therefore, the equation of this tangent plane is

$$\begin{aligned} \langle 5,2,6\rangle \cdot \langle x-1, y-0, z-1\rangle = 0 &\Longrightarrow 5(x-1) + 2(y-0) + 6(z-1) = 0 \\ &\Longrightarrow 5x + 2y + 6z - 11 = 0. \end{aligned}$$

Problem 4. [6+6 points] The temperature of a thin plate is given by the function

$$T(x,y) = x^2 - xy + y^2 - 6x + 2.$$

(i) Find the critical points of T and determine their type.

Solution. We have

$$\nabla T = \langle T_x, T_y\rangle = \langle 2x - y - 6, -x + 2y\rangle = \langle 0,0\rangle \Longrightarrow \begin{cases} 2x - y - 6 &= 0 \\ -x + 2y &= 0 \end{cases}$$

which has the unique solution $x = 4, y = 2$. It follows that $(4, 2)$ is the only critical point of T.

To determine the type of this critical point, we use the 2nd derivative test. We have

$$A = T_{xx}(4, 2) = 2, \;\; B = T_{xy}(4, 2) = T_{yx}(4, 2) = -1, \;\; C = T_{yy}(4, 2) = 2.$$

Hence the Hessian matrix H at $(4, 2)$ takes the form

$$H = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Since $\det(H) = 3 > 0$ and $A = 2 > 0$, the 2nd derivative test shows that $(4, 2)$ is a local minimum.

(ii) Find the maximum and minimum temperature over the rectangular region $0 \leq x \leq 5$, $0 \leq y \leq 3$.

Solution. We compute the maximum and minimum temperature along the boundary of this rectangle and compare it with the temperature at the critical point $(4, 2)$ inside this rectangle. We break up the boundary into four segments $S_1, \ldots, S_4$ as in the next page figure and study T on each segment separately.

- S_1: $0 \leq x \leq 5, y = 0$. Here T reduces to

$$g(x) = T(x, 0) = x^2 - 6x + 2, \qquad 0 \leq x \leq 5.$$

We have $g'(x) = 2x - 6$ so $x = 3$ is the only critical point of g. We have $g(0) = 2$, $g(3) = -7$ and $g(5) = -3$, so

$$\text{maximum of } T \text{ on } S_1 = f(0, 0) = 2$$
$$\text{minimum of } T \text{ on } S_1 = f(3, 0) = -7.$$

- S_2: $0 \leq x \leq 5, y = 3$. Here T reduces to

$$g(x) = T(x, 3) = x^2 - 9x + 11, \qquad 0 \leq x \leq 5.$$

We have $g'(x) = 2x - 9$ so $x = 9/2$ is the only critical point of g. We have $g(0) = 11$, $g(9/2) = -37/4$ and $g(5) = -9$, so

$$\text{maximum of } T \text{ on } S_2 = f(0, 3) = 11$$
$$\text{minimum of } T \text{ on } S_2 = f(9/2, 3) = -37/4.$$

- S_3: $0 \leq y \leq 3, x = 0$. Here T reduces to

$$g(y) = T(0, y) = y^2 + 2, \qquad 0 \leq y \leq 3.$$

We have $g'(y) = 2y$ so $y = 0$ is the only critical point of g. We have $g(0) = 2$, $g(3) = 11$, so

$$\text{maximum of } T \text{ on } S_3 = f(0, 3) = 11$$
$$\text{minimum of } T \text{ on } S_3 = f(0, 0) = 2.$$

- S_4: $0 \leq y \leq 3, x = 5$. Here T reduces to

$$g(y) = T(5, y) = y^2 - 5y - 3, \qquad 0 \leq y \leq 3.$$

We have $g'(y) = 2y - 5$ so $y = 5/2$ is the only critical point of g. We have $g(0) = -3$, $g(5/2) = -37/4$ and $g(3) = -9$, so

$$\text{maximum of } T \text{ on } S_4 = f(5,0) = -3$$
$$\text{minimum of } T \text{ on } S_4 = f(5,5/2) = -37/4.$$

Finally, considering these values together with the temperature

$$T(4,2) = -10$$

at the critical point, we conclude that

$$\text{maximum of } T \text{ on the rectangle} = f(0,3) = 11$$
$$\text{minimum of } T \text{ on the rectangle} = f(4,2) = -10.$$