

## Math 201 Final Exam Solutions

**Problem 1.** [7 points] Let  $\mathbf{a} = \langle \sqrt{6}, 1, -1 \rangle$  and  $\mathbf{b} = \langle \sqrt{6}, -1, 1 \rangle$ .

(i) Find the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

*Solution.* If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{6 - 1 - 1}{\sqrt{8} \sqrt{8}} = \frac{4}{8} = \frac{1}{2}.$$

Hence

$$\theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} = 60^\circ.$$

(ii) Find a vector  $\mathbf{v}$  which is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$  and has length 3.

*Solution.* The cross product  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & 1 & -1 \\ \sqrt{6} & -1 & 1 \end{vmatrix} \\ &= (1 - 1)\mathbf{i} - (\sqrt{6} + \sqrt{6})\mathbf{j} + (-\sqrt{6} - \sqrt{6})\mathbf{k} = -2\sqrt{6}(\mathbf{j} + \mathbf{k}). \end{aligned}$$

However, the length of the vector  $\mathbf{a} \times \mathbf{b}$  is

$$|\mathbf{a} \times \mathbf{b}| = 2\sqrt{6} |\mathbf{j} + \mathbf{k}| = 2\sqrt{6} \sqrt{2} = 2\sqrt{12}.$$

Thus, to obtain  $\mathbf{v}$ , it suffices to scalar multiply  $\mathbf{a} \times \mathbf{b}$  by  $3/(2\sqrt{12})$ :

$$\mathbf{v} = \frac{3}{2\sqrt{12}} \mathbf{a} \times \mathbf{b} = -\frac{3}{\sqrt{2}}(\mathbf{j} + \mathbf{k}).$$

Note that the vector  $(3/\sqrt{2})(\mathbf{j} + \mathbf{k})$  also satisfies the required conditions.

**Problem 2.** [9 points] Find the critical points of the function

$$f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$$

and determine their type.

*Solution.* We have

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 12x - 6x^2 + 6y, 6y + 6x \rangle,$$

so the critical points of  $f$  are the solutions of the system of equations

$$\begin{cases} 12x - 6x^2 + 6y = 0 \\ 6y + 6x = 0. \end{cases}$$

The second equation gives  $y = -x$ . Substituting this into the first gives  $6x - 6x^2 = 0$  which has the solutions  $x = 0$  and  $x = 1$ . Thus, the critical points are  $(0, 0)$  and  $(1, -1)$ .

To determine the type of these critical points, we use the 2nd derivative test. We have

$$f_{xx} = 12 - 12x, \quad f_{xy} = f_{yx} = 6, \quad f_{yy} = 6.$$

Hence the Hessian matrix has the form

$$H(x, y) = \begin{bmatrix} 12 - 12x & 6 \\ 6 & 6 \end{bmatrix}.$$

At the critical point  $(0,0)$ , we have

$$H(0,0) = \begin{bmatrix} 12 & 6 \\ 6 & 6 \end{bmatrix}.$$

Since  $\det(H) = 36 > 0$  and  $f_{xx} = 12 > 0$ , it follows that  $(0,0)$  is a local minimum. Similarly, at the critical point  $(1, -1)$ , we have

$$H(1, -1) = \begin{bmatrix} 0 & 6 \\ 6 & 6 \end{bmatrix}.$$

Since  $\det(H) = -36 < 0$ , it follows that  $(1, -1)$  is a saddle point.

**Problem 3.** [10 points] At what point(s) on the surface

$$2x^2 - y^2 + 3z^2 = 36$$

is the tangent plane orthogonal to the line  $3x = 6y = 2z$ ?

*Solution.* Let  $f(x, y, z) = 2x^2 - y^2 + 3z^2$  so the given surface is the level set  $f = 36$ . We know that at each point of this surface the gradient  $\nabla f$  is normal to the tangent plane. That means we should be looking for the points on this surface at which  $\nabla f$  is parallel to the line  $3x = 6y = 2z$ .

The symmetric equation of this line can be rewritten as

$$\frac{x}{\frac{1}{3}} = \frac{y}{\frac{1}{6}} = \frac{z}{\frac{1}{2}},$$

which shows that the line is in the direction of the vector  $\langle \frac{1}{3}, \frac{1}{6}, \frac{1}{2} \rangle$ , or to get rid of these fractions, the vector

$$\mathbf{v} = \langle 2, 1, 3 \rangle.$$

Thus, we are looking for the points on this surface at which  $\nabla f$  is parallel to  $\mathbf{v}$ . This happens when  $\nabla f = c\mathbf{v}$  for some scalar  $c$ :

$$\nabla f = c\mathbf{v} \implies \langle 4x, -2y, 6z \rangle = c\langle 2, 1, 3 \rangle \implies x = \frac{c}{2}, y = -\frac{c}{2}, z = \frac{c}{2}.$$

Since  $(x, y, z)$  belongs to the level set  $f = 36$ , we have

$$\begin{aligned} 2x^2 - y^2 + 3z^2 = 36 &\implies 2\left(\frac{c}{2}\right)^2 - \left(-\frac{c}{2}\right)^2 + 3\left(\frac{c}{2}\right)^2 = 36 \\ &\implies c^2\left(\frac{1}{2} - \frac{1}{4} + \frac{3}{4}\right) = 36 \\ &\implies c^2 = 36 \\ &\implies c = \pm 6. \end{aligned}$$

Thus, the points in question are

$$(3, -3, 3) \quad \text{and} \quad (-3, 3, -3).$$

**Problem 4.** [10 points] Evaluate the double integral

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \frac{x+y}{x^2+y^2} dy dx.$$

*Solution.* The bounds of the iterated integral show that the region of integration is

$$\{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\},$$

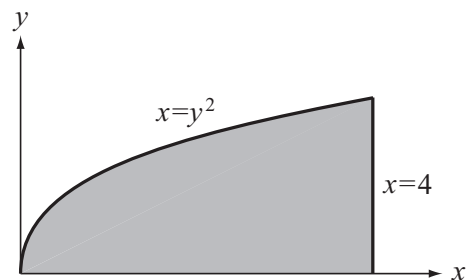
which is the upper semidisk of radius 1 centered at the origin. In polar coordinates, this is simply described as

$$\{(r, \theta) : 0 \leq \theta \leq \pi, 0 \leq r \leq 1\}.$$

Hence, by changing to polar coordinates, the integral becomes

$$\begin{aligned} \int_0^\pi \int_0^1 \frac{r \cos \theta + r \sin \theta}{r^2} r dr d\theta &= \int_0^\pi \int_0^1 (\cos \theta + \sin \theta) dr d\theta \\ &= \int_0^\pi (\cos \theta + \sin \theta) [r]_0^1 d\theta \\ &= \int_0^\pi (\cos \theta + \sin \theta) d\theta \\ &= [\sin \theta - \cos \theta]_0^\pi \\ &= 2. \end{aligned}$$

**Problem 5.** [12 points] Find the center of mass of the uniform thin plate bounded by the parabola  $x = y^2$ , the  $x$ -axis, and the line  $x = 4$ .



*Solution.* Since the plate  $D$  has uniform density, its center of mass  $(\bar{x}, \bar{y})$  is given by

$$\bar{x} = \frac{\iint_D x dA}{\iint_D dA} \quad \text{and} \quad \bar{y} = \frac{\iint_D y dA}{\iint_D dA}.$$

In computing these double integrals,  $D$  can be considered either as a region of type I (slicing vertically) or type II (slicing horizontally), but perhaps it would be a little easier

to consider it type II to avoid working with square roots. We have

$$\begin{aligned}\iint_D dA &= \int_0^2 \int_{y^2}^4 dx dy \\ &= \int_0^2 [x]_{y^2}^4 dy \\ &= \int_0^2 (4 - y^2) dy \\ &= [4y - \frac{1}{3}y^3]_0^2 \\ &= 8 - \frac{8}{3} = \frac{16}{3}.\end{aligned}$$

Similarly,

$$\begin{aligned}\iint_D x dA &= \int_0^2 \int_{y^2}^4 x dx dy \\ &= \int_0^2 [\frac{1}{2}x^2]_{y^2}^4 dy \\ &= \frac{1}{2} \int_0^2 (16 - y^4) dy \\ &= \frac{1}{2} [16y - \frac{1}{5}y^5]_0^2 \\ &= \frac{1}{2} [32 - \frac{32}{5}] = \frac{64}{5}.\end{aligned}$$

Finally,

$$\begin{aligned}\iint_D y dA &= \int_0^2 \int_{y^2}^4 y dx dy \\ &= \int_0^2 [xy]_{y^2}^4 dy \\ &= \int_0^2 (4y - y^3) dy \\ &= [2y^2 - \frac{1}{4}y^4]_0^2 \\ &= 8 - 4 = 4.\end{aligned}$$

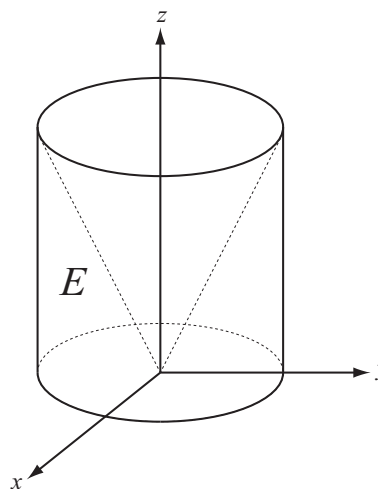
It follows that

$$\bar{x} = \frac{\frac{64}{5}}{\frac{16}{3}} = \frac{12}{5} \quad \text{and} \quad \bar{y} = \frac{4}{\frac{16}{3}} = \frac{3}{4}.$$

**Problem 6.** [12 points] Evaluate the triple integral

$$\iiint_E z \, dV,$$

where  $E$  is the solid that lies inside the cylinder  $x^2 + y^2 = 3$ , above the plane  $z = 0$ , and below the cone  $z^2 = 4x^2 + 4y^2$ .



*Solution.* We use cylindrical coordinates  $(r, \theta, z)$  in which the cylinder  $x^2 + y^2 = 3$  is described by  $r = \sqrt{3}$  and the cone  $z^2 = 4x^2 + 4y^2$  is described by  $z = 2r$ . Thus,

$$\begin{aligned} \iiint_E z \, dV &= \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{2r} z \, r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \left[ \frac{1}{2} z^2 r \right]_0^{2r} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} 2r^3 \, dr \, d\theta \\ &= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\sqrt{3}} 2r^3 \, dr \right) \\ &= 2\pi \left[ \frac{1}{2} r^4 \right]_0^{\sqrt{3}} \\ &= 2\pi \cdot \frac{9}{2} \\ &= 9\pi. \end{aligned}$$