4. $\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dz\,dy\,dx = \int_0^1 \int_x^{2x} \left[xyz^2\right]_{z=0}^{z=y} dy\,dx = \int_0^1 \int_x^{2x} xy^3\,dy\,dx$

$= \int_0^1 \left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x} dx = \int_0^1 \frac{15}{4}x^5\,dx = \frac{5}{8}x^6\Big]_0^1 = \frac{5}{8}$

6. $\int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dx\,dy\,dz = \int_0^1 \int_0^z \left[xze^{-y^2}\right]_{x=0}^{x=y} dy\,dz = \int_0^1 \int_0^z yze^{-y^2}\,dy\,dz = \int_0^1 \left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z} dz$

$= \int_0^1 -\frac{1}{2}z\left(e^{-z^2} - 1\right) dz = \frac{1}{2}\int_0^1 \left(z - ze^{-z^2}\right) dz$

$= \frac{1}{2}\left[\frac{1}{2}z^2 + \frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}(1 + e^{-1} - 0 - 1) = \frac{1}{4e}$

8. $\iiint_E yz\cos(x^5)\,dV = \int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx = \int_0^1 \int_0^x \left[\frac{1}{2}yz^2\cos(x^5)\right]_{z=x}^{z=2x} dy\,dx$

$= \frac{1}{2}\int_0^1 \int_0^x 3x^2 y\cos(x^5)\,dy\,dx = \frac{1}{2}\int_0^1 \left[\frac{3}{2}x^2y^2\cos(x^5)\right]_{y=0}^{y=x} dx$

$= \frac{3}{4}\int_0^1 x^4\cos(x^5)\,dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20}\sin 1$

10. Here E is the region in the first octant that lies below the plane $2x + 2y + z = 4$ (and above the region in the xy-plane bounded by the lines $x = 0$, $y = 0$, $x + y = 2$). So

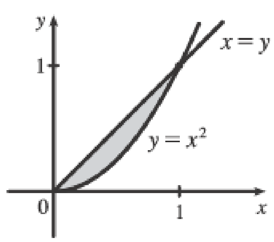
$\iiint_E y\,dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\,dz\,dy\,dx = \int_0^2 \int_0^{2-x} y(4 - 2x - 2y)\,dy\,dx = \int_0^2 \int_0^{2-x} (4y - 2xy - 2y^2)\,dy\,dx$

$= \int_0^2 \left[2y^2 - xy^2 - \frac{2}{3}y^3\right]_{y=0}^{y=2-x} dx = \int_0^2 \left[2(2-x)^2 - x(2-x)^2 - \frac{2}{3}(2-x)^3\right] dx$

$= \int_0^2 \left[(2-x)(2-x)^2 - \frac{2}{3}(2-x)^3\right] dx = \frac{1}{3}\int_0^2 (2-x)^3\,dx$

$= \frac{1}{3}\left[-\frac{1}{4}(2-x)^4\right]_0^2 = -\frac{1}{12}(0 - 16) = \frac{4}{3}$

14.



E is the solid above the region shown in the xy-plane and below the plane $z = x$. Thus,

$\iiint_E (x + 2y)\,dV = \int_0^1 \int_{x^2}^x \int_0^x (x + 2y)\,dz\,dy\,dx$

$= \int_0^1 \int_{x^2}^x (x^2 + 2yx)\,dy\,dx = \int_0^1 \left[x^2y + xy^2\right]_{y=x^2}^{y=x} dx$

$= \int_0^1 (2x^3 - x^4 - x^5)\,dx = \left[\frac{1}{2}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6\right]_0^1 = \frac{2}{15}$

20. The paraboloid $x = y^2 + z^2$ intersects the plane $x = 16$ in the circle $y^2 + z^2 = 16$, $x = 16$.

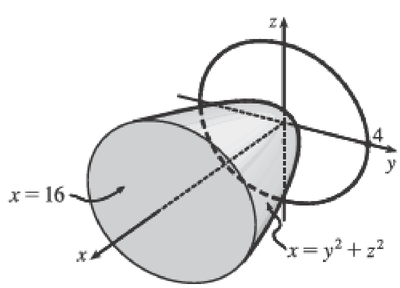
Thus, $E = \left\{(x, y, z) \mid y^2 + z^2 \le x \le 16,\ y^2 + z^2 \le 16\right\}$.

Let $D = \left\{(y, z) \mid y^2 + z^2 \le 16\right\}$. Then using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we have

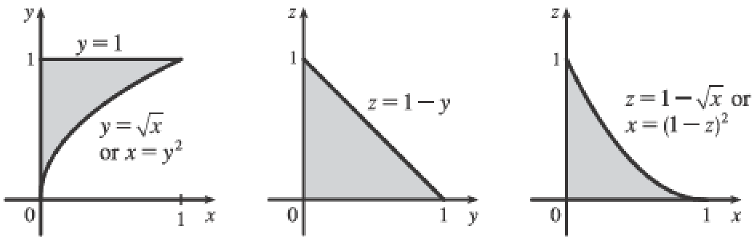
$V = \iint_D \left(\int_{y^2+z^2}^{16} dx\right) dA = \iint_D (16 - (y^2 + z^2))\,dA$

$= \int_0^{2\pi} \int_0^4 (16 - r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 (16r - r^3)\,dr$

$= \left[\theta\right]_0^{2\pi} \left[8r^2 - \frac{1}{4}r^4\right]_0^4 = 2\pi(128 - 64) = 128\pi$



31.



The diagrams show the projections of E on the xy-, yz-, and xz-planes. Therefore

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$
$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

38. $m = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y - y^2\right]dy\,dx$

$= \int_0^1\left[\frac{1}{2}(1-x)^3 - \frac{1}{3}(1-x)^3\right]dx = \frac{1}{6}\int_0^1(1-x)^3\,dx = \frac{1}{24}$

$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(x-x^2)y - xy^2\right]dy\,dx$

$= \int_0^1\left[\frac{1}{2}x(1-x)^3 - \frac{1}{3}x(1-x)^3\right]dx = \frac{1}{6}\int_0^1\left(x - 3x^2 + 3x^3 - x^4\right)dx = \frac{1}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = \frac{1}{120}$

$M_{xz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y^2 - y^3\right]dy\,dx$

$= \int_0^1\left[\frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4\right]dx = \frac{1}{12}\left[-\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{60}$

$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\frac{1}{2}y(1-x-y)^2\right]dy\,dx$

$= \frac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y - 2(1-x)y^2 + y^3\right]dy\,dx = \frac{1}{2}\int_0^1\left[\frac{1}{2}(1-x)^4 - \frac{2}{3}(1-x)^4 + \frac{1}{4}(1-x)^4\right]dx$

$= \frac{1}{24}\int_0^1(1-x)^4\,dx = -\frac{1}{24}\left[\frac{1}{5}(1-x)^5\right]_0^1 = \frac{1}{120}$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(\frac{1}{5},\frac{2}{5},\frac{1}{5}\right)$.

5. Since $r = 3$, $x^2 + y^2 = 9$ and the surface is a circular cylinder with radius 3 and axis the z-axis.

6. Whether spherical or cylindrical coordinates, since $\theta = \frac{\pi}{3}$ the surface is a half-plane including the z-axis and intersecting the xy-plane in the half-line $y = \sqrt{3}\,x$, $x > 0$.

17. In cylindrical coordinates, E is given by $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^4 r^2\,dr\int_{-5}^4 dz$$
$$= \left[\theta\right]_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^4\left[z\right]_{-5}^4 = (2\pi)\left(\frac{64}{3}\right)(9) = 384\pi$$

18. The paraboloid $z = 1 - x^2 - y^2$ intersects the xy-plane in the circle $x^2 + y^2 = r^2 = 1$ or $r = 1$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1, 0 \le z \le 1 - r^2\right\}$. Thus

$$\iiint_E (x^3 + xy^2)\, dV = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} (r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)\, r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^4\cos\theta\left[\, z\,\right]_{z=0}^{z=1-r^2} dr\, d\theta = \int_0^{\pi/2}\int_0^1 r^4(1 - r^2)\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2}\cos\theta\left[\tfrac{1}{5}r^5 - \tfrac{1}{7}r^7\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2}\tfrac{2}{35}\cos\theta\, d\theta = \tfrac{2}{35}\left[\sin\theta\right]_0^{\pi/2} = \tfrac{2}{35}$$

20. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\cos\theta + r\sin\theta + 5$ and the cylinders $r = 2$ and $r = 3$, so E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 2 \le r \le 3, 0 \le z \le r\cos\theta + r\sin\theta + 5\}$. Thus

$$\iiint_E x\, dV = \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 5}(r\cos\theta)\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_2^3 (r^2\cos\theta)\left[\, z\,\right]_{z=0}^{z=r\cos\theta + r\sin\theta+5} dr\, d\theta$$

$$= \int_0^{2\pi}\int_2^3 (r^2\cos\theta)(r\cos\theta + r\sin\theta + 5)\, dr\, d\theta = \int_0^{2\pi}\int_2^3 (r^3(\cos^2\theta + \cos\theta\sin\theta) + 5r^2\cos\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\tfrac{1}{4}r^4(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3}r^3\cos\theta\right]_{r=2}^{r=3} d\theta$$

$$= \int_0^{2\pi}\left[\left(\tfrac{81}{4} - \tfrac{16}{4}\right)(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3}(27 - 8)\cos\theta\right] d\theta$$

$$= \int_0^{2\pi}\left(\tfrac{65}{4}\left(\tfrac{1}{2}(1 + \cos 2\theta) + \cos\theta\sin\theta\right) + \tfrac{95}{3}\cos\theta\right) d\theta = \left[\tfrac{65}{8}\theta + \tfrac{65}{16}\sin 2\theta + \tfrac{65}{8}\sin^2\theta + \tfrac{95}{3}\sin\theta\right]_0^{2\pi} = \tfrac{65}{4}\pi$$

28. The region of integration is the region above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. Also, we have $-3 \le x \le 3$ with $0 \le y \le \sqrt{9 - x^2}$ which describes the upper half of a circle of radius 3 in the xy-plane centered at $(0, 0)$. Thus,

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\, dz\, dy\, dx = \int_0^{\pi}\int_0^3\int_0^{9-r^2}\sqrt{r^2}\, r\, dz\, dr\, d\theta = \int_0^{\pi}\int_0^3\int_0^{9-r^2} r^2\, dz\, dr\, d\theta$$

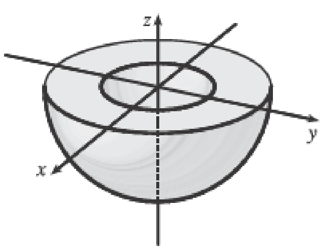
$$= \int_0^{\pi}\int_0^3 r^2\left(9 - r^2\right) dr\, d\theta = \int_0^{\pi} d\theta \int_0^3\left(9r^2 - r^4\right) dr$$

$$= \left[\,\theta\,\right]_0^{\pi}\left[3r^3 - \tfrac{1}{5}r^5\right]_0^3 = \pi\left(81 - \tfrac{243}{5}\right) = \tfrac{162}{5}\pi$$

5. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z-axis.

6. Since $\rho = 3$, $x^2 + y^2 + z^2 = 9$ and the surface is a sphere with center the origin and radius 3.

18.



The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, 0 \le \theta \le 2\pi, \pi/2 \le \phi \le \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy-plane.

$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi}\sin\phi\, d\phi \int_1^2 \rho^2\, d\rho$$

$$= \left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi\right]_{\pi/2}^{\pi}\left[\tfrac{1}{3}\rho^3\right]_1^2$$

$$= 2\pi(1)\left(\tfrac{7}{3}\right) = \tfrac{14\pi}{3}$$

22. In spherical coordinates, H is represented by $\left\{(\rho,\theta,\phi)\mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\iiint_H (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^1\rho^4\,d\rho$$

$$= \left[\,\theta\,\right]_0^{2\pi}\left[-\cos\phi+\tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^1 = \tfrac{4\pi}{15}$$

23. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi)\mid 1\le\rho\le 2, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_1^2\rho^3\,d\rho$$

$$= \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2}\left[\,\theta\,\right]_0^{\pi/2}\left[\tfrac{1}{4}\rho^4\right]_1^2 = \left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right)\left(\tfrac{15}{4}\right) = \tfrac{15\pi}{16}$$

36. The region of integration is the solid sphere $x^2+y^2+z^2\le a^2$, so $0\le\theta\le 2\pi$, $0\le\phi\le\pi$, and $0\le\rho\le a$. Also $x^2z+y^2z+z^3=(x^2+y^2+z^2)z=\rho^2z=\rho^3\cos\phi$, so the integral becomes

$$\int_0^{\pi}\int_0^{2\pi}\int_0^a\left(\rho^3\cos\phi\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin\phi\cos\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a\rho^5\,d\rho = \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi}\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{6}\rho^6\right]_0^a = 0$$