8. $\iint_R (x+y)\,dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$

$$= \left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right)\left(\int_1^2 r^2\,dr\right) = \left[\sin\theta-\cos\theta\right]_{\pi/2}^{3\pi/2}\left[\tfrac13 r^3\right]_1^2$$

$$= (-1-0-1+0)\left(\tfrac83-\tfrac13\right) = -\tfrac{14}{3}$$

10. $\iint_R \sqrt{4-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 \sqrt{4-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 r\sqrt{4-r^2}\,dr\right)$

$$= \left[\,\theta\,\right]_{-\pi/2}^{\pi/2}\left[-\tfrac12\cdot\tfrac23(4-r^2)^{3/2}\right]_0^2 = \left(\tfrac{\pi}{2}+\tfrac{\pi}{2}\right)\left(-\tfrac13(0-4^{3/2})\right) = \tfrac83\pi$$

12. $\iint_R ye^x\,dA = \int_0^{\pi/2}\int_0^5 (r\sin\theta)\,e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr$. First we integrate $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta$:

Let $u = r\cos\theta \quad\Rightarrow\quad du = -r\sin\theta\,d\theta$, and $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta = \int_{u=r}^{u=0} -r\,e^u\,du = -r[e^0 - e^r] = re^r - r$.

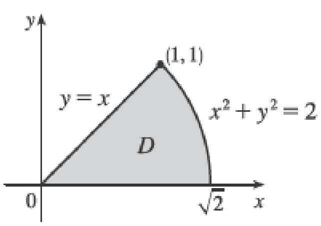
Then $\int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r - r)\,dr = \left[re^r - e^r - \tfrac12 r^2\right]_0^5 = 4e^5 - \tfrac{23}{2}$, where we integrated by parts in the first term.

16. The sphere $x^2+y^2+z^2=16$ intersects the xy-plane in the circle $x^2+y^2=16$, so

$$V = 2\iint_{4\le x^2+y^2\le 16}\sqrt{16-x^2-y^2}\,dA \quad \text{[by symmetry]} \quad = 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi} d\theta\int_2^4 r(16-r^2)^{1/2}dr$$

$$= 2\left[\,\theta\,\right]_0^{2\pi}\left[-\tfrac13(16-r^2)^{3/2}\right]_2^4 = -\tfrac23(2\pi)(0-12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt3\,\pi$$

25.



$$\int_0^{\pi/4}\int_0^{\sqrt2}(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/4}(\cos\theta+\sin\theta)\,d\theta\int_0^{\sqrt2} r^2\,dr$$

$$= \left[\sin\theta-\cos\theta\right]_0^{\pi/4}\left[\tfrac13 r^3\right]_0^{\sqrt2}$$

$$= \left[\tfrac{\sqrt2}{2}-\tfrac{\sqrt2}{2}-0+1\right]\cdot\tfrac13\left(2\sqrt2-0\right) = \tfrac{2\sqrt2}{3}$$

27. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x,y) \mid x^2+y^2 \le 400\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20)=2$ to $f(0,20)=7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z-7=\frac{1}{8}(y-20) \quad \Rightarrow \quad z=\frac{1}{8}y+\frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y)=\frac{1}{8}y+\frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D=\{(r,\theta) \mid 0\le r\le 20, 0\le\theta\le 2\pi\}$ and substituting $x=r\cos\theta$, $y=r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta+\tfrac{9}{2}\right)r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta+\tfrac{9}{4}r^2\right]_{r=0}^{r=20}d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta+900\right)d\theta$$
$$= \left[-\tfrac{1000}{3}\cos\theta+900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi \approx 5655\ \text{ft}^3$ of water.

4. $m = \iint_D \rho(x,y)\,dA = \int_0^a\int_0^b cxy\,dy\,dx = c\int_0^a x\,dx\int_0^b y\,dy = c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{2}y^2\right]_0^b = \frac{1}{4}a^2b^2c$,

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^a\int_0^b cx^2y\,dy\,dx = c\int_0^a x^2\,dx\int_0^b y\,dy = c\left[\frac{1}{3}x^3\right]_0^a\left[\frac{1}{2}y^2\right]_0^b = \frac{1}{6}a^3b^2c$, and

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^a\int_0^b cxy^2\,dy\,dx = c\int_0^a x\,dx\int_0^b y^2\,dy = c\left[\frac{1}{2}x^2\right]_0^a\left[\frac{1}{3}y^3\right]_0^b = \frac{1}{6}a^2b^3c$.

Hence, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\frac{2}{3}a,\frac{2}{3}b\right)$.

6. $m = \int_0^1\int_y^{4-3y} x\,dx\,dy = \int_0^1\left[\frac{1}{2}(4-3y)^2-\frac{1}{2}y^2\right]dy = \left[-\frac{1}{18}(4-3y)^3-\frac{1}{6}y^3\right]_0^1 = \frac{10}{3}$,

$M_y = \int_0^1\int_y^{4-3y} x^2\,dx\,dy = \int_0^1\left[\frac{1}{3}(4-3y)^3-\frac{1}{3}y^3\right]dy = \left[-\frac{1}{36}(4-3y)^4-\frac{1}{12}y^4\right]_0^1 = 7$,

$M_x = \int_0^1\int_y^{4-3y} xy\,dx\,dy = \int_0^1\left[\frac{1}{2}y(4-3y)^2-\frac{1}{2}y^3\right]dy = \int_0^1(8y-12y^2+4y^3)\,dy = 1$.

Hence $m=\frac{10}{3}$, $(\overline{x},\overline{y}) = (2.1, 0.3)$.

8. $m = \int_0^1\int_0^{\sqrt{x}} x\,dy\,dx = \int_0^1 x\big[y\big]_{y=0}^{y=\sqrt{x}}dx = \int_0^1 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big]_0^1 = \frac{2}{5}$,

$M_y = \int_0^1\int_0^{\sqrt{x}} x^2\,dy\,dx = \int_0^1 x\big[y\big]_{y=0}^{y=\sqrt{x}}dx = \int_0^1 x^{5/2}\,dx = \frac{2}{7}x^{7/2}\Big]_0^1 = \frac{2}{7}$,

$M_x = \int_0^1\int_0^{\sqrt{x}} yx\,dy\,dx = \int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}dx = \frac{1}{2}\int_0^1 x^2\,dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$.

Hence $m=\frac{2}{5}$, $(\overline{x},\overline{y}) = \left(\frac{2/7}{2/5},\frac{1/6}{2/5}\right) = \left(\frac{5}{7},\frac{5}{12}\right)$.

12. $\rho(x,y) = k(x^2+y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^3\,dr\,d\theta = \frac{\pi}{8}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^4\cos\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{5}k\big[\sin\theta\big]_0^{\pi/2} = \frac{1}{5}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^4\sin\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\sin\theta\,d\theta = \frac{1}{5}k\big[-\cos\theta\big]_0^{\pi/2} = \frac{1}{5}k$.

Hence $(\overline{x},\overline{y}) = \left(\frac{8}{5\pi},\frac{8}{5\pi}\right)$.