Math 202 Second Midterm Review Sheet

4/6/2024

Here is a list of the material that the second midterm is based on. In addition to the sample problems below, you can review your lecture notes and the examples discussed in class, the worked-out examples in the book, and the homework problems and solutions available on WebAssign.

- Vector fields and their trajectories, gradient as a vector field

- Line integrals of scalar functions: If C is a smooth curve parametrized by $\mathbf{r}(t)$ for $t \in [a, b]$ and f is a continuous scalar function on C, then
$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt.$$
The case $f = 1$ gives the arc-length of C. The *average value of f over C* is defined by
$$\operatorname{avg}_C(f) = \frac{\int_C f\, ds}{\int_C ds} = \frac{\int_C f\, ds}{\operatorname{length}(C)}.$$

- Line integrals of vector fields: If C is a smooth curve parametrized by $\mathbf{r}(t)$ for $t \in [a, b]$ and $\mathbf{F}$ is a continuous vector field on C, then
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$
In particular, if $\mathbf{F}$ is everywhere normal to C, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

- Classical notation for line integrals: Write $\mathbf{r}(t) = x(t)\, \mathbf{i} + y(t)\, \mathbf{j} + z(t)\, \mathbf{k}$ and $F = P\, \mathbf{i} + Q\, \mathbf{j} + R\, \mathbf{k}$. Then
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz.$$

- Line integrals of vector fields (unlike scalar functions) are orientation sensitive: Reversing the orientation of C will negate the line integral.

- The fundamental theorem of calculus for gradient vector fields:
$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$
Thus, gradients vector fields are *conservative*, that is, their line integral only depends on the initial and end points of the curve. Equivalently, their line integral over any closed curve is zero.

- Conversely, every conservative vector field $\mathbf{F}$ in a region D is a gradient vector field, so there is a scalar function $f : D \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$ everywhere in D. The *potential function* f is unique up to an additive constant. In practice, f can be obtained by taking suitable anti-derivatives of the components of $\mathbf{F}$.

- Definition of curl and divergence of a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$:

$$\begin{aligned} \operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ \operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} &= P_x + Q_y + R_z. \end{aligned}$$

- Two important results: For any smooth scalar function f and vector field $\mathbf{F}$,

$$\operatorname{curl}(\operatorname{grad}(f)) = 0 \qquad \text{and} \qquad \operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

- How to recognize gradient vector fields: Suppose $\mathbf{F}$ is a smooth vector field defined in a region D. If $\mathbf{F}$ is a gradient vector field, then $\operatorname{curl}(\mathbf{F}) = 0$. Conversely, if $\operatorname{curl}(\mathbf{F}) = 0$, then $\mathbf{F}$ is a gradient vector field provided that D is *simply connected* (i.e., every loop in D can be continuously shrunk to a point all the while remaining in D).

  If $D \subset \mathbb{R}^3$ and $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$, the condition $\operatorname{curl}(\mathbf{F}) = 0$ means $R_y = Q_z$, $R_x = P_z$, $Q_x = P_y$. If $D \subset \mathbb{R}^2$ and $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, the condition $\operatorname{curl}(\mathbf{F}) = 0$ means $Q_x = P_y$.

- Green's theorem: Suppose $D \subset \mathbb{R}^2$ is a simply connected region bounded by the piecewise smooth positively oriented closed curve C. Then, for every smooth vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ in D,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dx\,dy.$$

  Special case:

$$\operatorname{area}(D) = -\int_C y\,dx = \int_C x\,dy = \frac{1}{2}\int_C (-y\,dx + x\,dy).$$

Practice Problems

1. Consider the vector field $\mathbf{F}(x,y) = 3y\,\mathbf{j}$ in the plane.

   (i) Draw (by hand) a rough plot of this vector field. Make a reasonable guess as to what the trajectories of $\mathbf{F}$ should look like.

(ii) Find a formula for the trajectory $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ of $\mathbf{F}$ which initially starts at $x_0\,\mathbf{i} + y_0\,\mathbf{j}$, i.e., $x(0) = x_0, y(0) = y_0$.

2. Evaluate $\int_C f\,ds$, where $f(x,y,z) = \exp\sqrt{z}$ and C is the curve parametrized by $\mathbf{r}(t) = \mathbf{i} + 2\mathbf{j} + t^2\mathbf{k}$ for $0 \le t \le 1$.

3. Evaluate:

(i) $\int_C y^2\,dx + 2xy\,dy$, where C is the positively oriented boundary of the square $[-1,1] \times [-1,1]$.

(ii) $\int_C x\,dx + y\,dy + (xz - y)\,dz$, where C is the oriented line segment from $(0,0,0)$ to $(1,2,4)$.

4. Verify that the vector field

$$\mathbf{F}(x,y,z) = (x^2 + y + z)\,\mathbf{i} + (x + y^2 + z)\,\mathbf{j} + (x + y + z^2)\,\mathbf{k}$$

in $\mathbb{R}^3$ is conservative. Then find an explicit formula for a potential for $\mathbf{F}$.

5. Let C_1 and C_4 denote the positively oriented circles of radii 1 and 4 centered at the origin. Find

$$\int_{C_4} (x^2y\,dx - xy^2\,dy) - \int_{C_1} (x^2y\,dx - xy^2\,dy).$$

6. The *Laplacian* of a smooth function $f(x,y,z)$ is defined by

$$\Delta f = f_{xx} + f_{yy} + f_{zz}.$$

When $\Delta f = 0$, we say that f is *harmonic*.

(i) Verify the identity $\Delta f = \mathrm{div}(\mathrm{grad}\, f)$.

(ii) Show by direct computation that $f(x,y,z) = (x^2 + y^2 + z^2)^{-1/2}$ is harmonic in $\mathbb{R}^3 - \{(0,0,0)\}$.