[0, 1], and for $p \in [0,1]$ let A_p denote the ideal of C consisting of those functions in C that vanish at p. The fact that the maximal ideals of C are precisely the ideals of the form A_p for some $p \in [0,1]$ provides an interesting example in a first undergraduate-level course in abstract algebra [**2**, p. 139] and a glimpse at one of the basic relationships between algebra and analysis. The feature of [0, 1] that is essential in the proof of the more difficult implication of this characterization (that every maximal ideal is of the form A_p) is its compactness [**1**, p. 58]. However, the compactness of [0, 1] may be exploited in an equivalent form: the usual metric on [0, 1] is complete and totally bounded. Applying the concepts of Cauchy sequences and completeness, which are of fundamental importance in calculus, to characterize the maximal ideals of C seems quite appropriate at the undergraduate level.

The purpose of this note is to present a proof that (1) treats the completeness of [0, 1] as its essential ingredient and (2) emphasizes the interplay between the algebraic properties of C and the behavior of the continuous functions on [0, 1].

THEOREM. *If A is a maximal ideal of C, then there is a point $p \in [0,1]$ such that $A = A_p$.*

Proof. Note for reference that each member of any proper ideal of C must vanish at some point of [0, 1]. (A member of C that never vanishes on [0, 1] is an invertible member of C.) Also note that if the sum of two nonnegative functions vanishes at some point then each of the two functions vanishes at that point.

Let n be a positive integer and for $1 \leqslant i \leqslant n$ let h_i be a nonnegative member of C such that $h_i(x) = 0$ if and only if $(i-1)/n \leqslant x \leqslant i/n$. Since C is a commutative ring with identity, every maximal ideal of C is a prime ideal [**2**, p. 167]. So A is a prime ideal, and, since $h_1 \cdot h_2 \cdots h_n = 0$, $h_i \in A$ for some i. Rename this h_i as f_n.

In this way for each positive integer n we can construct a nonnegative function $f_n \in A$ whose zero set is a closed interval $[a_n, b_n] \subseteq [0,1]$ of length $1/n$ (i.e., $f_n(x) = 0$ if and only if $x \in [a_n, b_n]$). Now if m and n are two positive integers then $f_m + f_n \in A$. So there is a point $x_{m,n} \in [0,1]$ for which $f_m(x_{m,n}) + f_n(x_{m,n}) = 0$, and hence $x_{m,n} \in [a_m, b_m] \cap [a_n, b_n]$. Using these facts and the completeness of [0, 1], we can easily show that the sequences $\{a_n\}$ and $\{b_n\}$ are Cauchy and have a common limit $p \in [0,1]$.

Now suppose $f \in A$. For each positive integer n, $f^2 + f_n \in A$. So there is a point $x_n \in [0,1]$ for which $f(x_n)^2 + f_n(x_n) = 0$, and hence $x_n \in [a_n, b_n]$ and $f_n(x_n) = 0$. Since $x_n \to p$, $f(p) = 0$ by the continuity of f. Thus, $f \in A_p$.

Therefore, $A \subseteq A_p$, and, since A is a maximal ideal, $A = A_p$.

Note. The total boundedness of [0, 1] is used in the proof in a completely transparent fashion. Let X be any complete and totally bounded metric space, let $C(X)$ be the ring of continuous real-valued functions on X, and let A be a maximal ideal of $C(X)$. Then we can use the total boundedness of X to construct nonnegative functions $f_n \in A$ with zero sets Z_n of diameter less than or equal to $1/n$. It can be shown that there is a point $p \in X$ which is the common limit of all sequences whose nth term lies in Z_n. The remainder of the proof translates directly to obtain $A = A_p$.

References

1. L. Gillman and M. Jerison, Rings of Continuous Functions, D. Van Nostrand, Princeton, N.J., 1960.
2. I. N. Herstein, Topics in Algebra, 2nd ed., Wiley, New York, 1975.

THE PLANIMETER AS AN EXAMPLE OF GREEN'S THEOREM

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The purpose of this note is to describe the use of Green's Theorem to explain the working of the polar (or Amsler's) planimeter in computing the area of a plane figure. While there are other, more elementary, descriptions of the planimeter (see references [**1**] and [**2**]), this description provides a substantial use of Green's Theorem and has proved interesting to typical second-year calculus classes.

An idealized form of the mechanics of the planimeter is shown in Fig. 1. Two arms OA and AB of fixed, unit length are attached at a pivot point A. In use, point O is fixed and point B is made to traverse the boundary of the region M, the area of which is to be determined. A wheel with counter is attached to A so that it rolls in a direction perpendicular to arm AB. The motion of B around the boundary of M causes a motion of A along the unit circle with center O. The component of the motion of A normal to AB causes the wheel to roll and the counter to record the distance traveled. The component of the motion of A along AB causes the wheel to slide. Thus, the distance traveled in the direction along AB is not recorded on the counter. We will see that the total rolling distance of the wheel, as recorded by the counter, is the area of M.

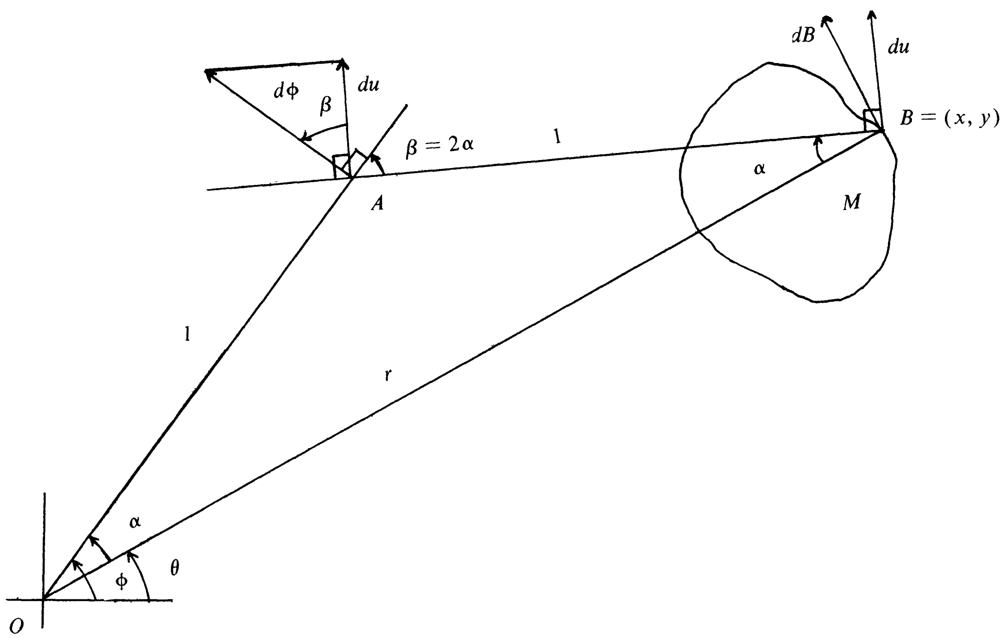

FIG. 1

We derive an integral formula for the total rolling distance of the wheel. Suppose B moves an infinitesimal distance dB along the boundary of M with du the component of dB perpendicular to AB. The resulting infinitesimal of rolling distance is then du and the total rolling distance is

$$\oint_{\partial M} du$$

where ∂M denotes the boundary of M.

Consider a fixed coordinate system at O. We shall write $du = P\,dx + Q\,dy$ and apply Green's Theorem to compute

$$\oint_{\partial M} du = \oint_{\partial M} P\,dx + Q\,dy = \iint_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Then by showing $\partial Q/\partial x - \partial P/\partial y = 1$ we shall have

$$\oint_{\partial M} du = \iint_M dx\,dy = \text{area of } M.$$

To find P and Q, let (x, y) denote the rectangular coordinates of B and (r, θ) denote its polar coordinates. Recall that

$$dr = \frac{x}{r}dx + \frac{y}{r}dy$$

and

$$d\theta = \frac{-y}{r^2}dx + \frac{x}{r^2}dy.$$

From the Law of Cosines applied to triangle OAB,

$$1 = 1 + r^2 - 2r\cos\alpha$$

so

$$\alpha = \cos^{-1}(r/2)$$

and

$$\phi = \theta + \alpha = \theta + \cos^{-1}(r/2).$$

Consequently

$$d\phi = d\theta - \frac{dr}{\sqrt{4 - r^2}}$$

and, upon our using the geometric observation that $\beta = 2\alpha$,

$$\begin{aligned} du &= \cos\beta\, d\phi = \cos 2\alpha\, d\phi = (2\cos^2\alpha - 1)\, d\phi = \left(\frac{r^2}{2} - 1\right) d\phi \\ &= \left(\frac{-y}{2} - \frac{rx}{2\sqrt{4-r^2}} + \frac{y}{r^2} + \frac{x}{r\sqrt{4-r^2}}\right) dx \\ &\quad + \left(\frac{x}{2} - \frac{ry}{2\sqrt{4-r^2}} - \frac{x}{r^2} + \frac{y}{r\sqrt{4-r^2}}\right) dy \\ &= P(x, y)\, dx + Q(x, y)\, dy. \end{aligned}$$

The computations of the partial derivatives are eased by noting that if $f(r)$ is any differentiable function of r then

$$y\frac{\partial f}{\partial x} = y\frac{df}{dr}\frac{x}{r} = \frac{xy}{r}\frac{df}{dr} = x\frac{\partial f}{\partial y}$$

so that

$$x\frac{\partial}{\partial y}\left[\frac{-r}{2\sqrt{4-r^2}} + \frac{1}{r\sqrt{4-r^2}}\right] = y\frac{\partial}{\partial x}\left[\frac{-r}{2\sqrt{4-r^2}} + \frac{1}{r\sqrt{4-r^2}}\right].$$

A direct computation shows

$$\frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) = \frac{x^2 - y^2}{r^4} = \frac{\partial}{\partial x}\left(-\frac{x}{r^2}\right).$$

Thus

$$\oint_M du = \iint_M \left(\frac{\partial\left(\frac{x}{2}\right)}{\partial x} - \frac{\partial\left(-\frac{y}{2}\right)}{\partial y}\right) dx\, dy = \iint_M dx\, dy$$

and so is equal to the area of M. It should be observed that the computation above demonstrates the fact that, in computing the line integral over a closed path, C,

$$\oint_C (F+G) = \oint_C F$$

for any conservative function G.

As a final remark to the instructor wishing to use this material in the classroom, obtain a planimeter and demonstrate or allow the students to use it to see what a simple mechanical device it is. To locate a planimeter, try your geography or civil engineering departments.

References

1. R. Courant, Differential and Integral Calculus, vol. 2, Interscience, New York, 1936.
2. Compensating Polar Planimeter Instruction Manual, Keuffel and Esser Co., New York, 1963.

PROBLEMS AND SOLUTIONS

*Send all **proposed** problems, in duplicate if possible, to Professor Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

SOLUTIONS OF PROBLEMS DEDICATED TO EMORY P. STARKE

Transformed Euler Differential Equation

S 28 [1980, 218]. *Proposed by David A. Sánchez, University of New Mexico.*

Find the general solution of the differential equation

$$x^3y'' + 2x^2 = (xy' - y)^2.$$

Solution by Deborah Frank Lockhart, Michigan Technological University. Let $y = -x\ln|v|$. Then $x^2v'' + 2xv' - 2v = 0$, which is an Euler equation with solution $v = \alpha x^{-2} + \beta x$. Hence

$$y = -x\ln|\alpha x^{-2} + \beta x|.$$

Note. Several solvers noted that absolute value signs can be dropped if the variables are complex. Some kept track of one parameter families of "singular" solutions "lost" in certain manipulations, but E. V. Norrset noted using the Ritt Theory of algebraic differential equations