Let k denote the nullity of T. If $v_1, \ldots, v_k$ are k independent solutions of the homogeneous system $T(x) = O$, and if b is one particular solution of the nonhomogeneous system $T(x) = c$, then the general solution of the nonhomogeneous system is

$$x = b + t_1v_1 + \cdots + t_kv_k ,$$

where $t_1, \ldots, t_k$ are arbitrary scalars.

This theorem does not tell us how to decide if a nonhomogeneous system has a particular solution b, nor does it tell us how to determine solutions $v_1, \ldots, v_k$ of the homogeneous system. It does tell us what to expect when the nonhomogeneous system has a solution. The following example, although very simple, illustrates the theorem.

EXAMPLE. The system $x + y = 2$ has for its associated homogeneous system the equation $x + y = 0$. Therefore, the null space consists of all vectors in V_2 of the form $(t, -t)$, where t is arbitrary. Since $(t, -t) = t(1, -1)$, this is a one-dimensional subspace of V_2 with basis $(1, -1)$. A particular solution of the nonhomogeneous system is $(0, 2)$. Therefore the general solution of the nonhomogeneous system is given by

$$(x, y) = (0, 2) + t(1, -1) \quad \text{or} \quad x = t, \quad y = 2 - t ,$$

where t is arbitrary.

16.18 Computation techniques

We turn now to the problem of actually computing the solutions of a nonhomogeneous linear system. Although many methods have been developed for attacking this problem, all of them require considerable computation if the system is large. For example, to solve a system of ten equations in as many unknowns can require several hours of hand computation, even with the aid of a desk calculator.

We shall discuss a widely-used method, known as the Gauss-Jordan elimination method, which is relatively simple and can be easily programmed for high-speed electronic computing machines. The method consists of applying three basic types of operations on the equations of a linear system:

(1) *Interchanging two equations;*

(2) *Multiplying all the terms of an equation by a nonzero scalar;*

(3) *Adding to one equation a multiple of another.*

Each time we perform one of these operations on the system we obtain a new system having exactly the same solutions. Two such systems are called *equivalent*. By performing these operations over and over again in a systematic fashion we finally arrive at an equivalent system which can be solved by inspection.

We shall illustrate the method with some specific examples. It will then be clear how the method is to be applied in general.

EXAMPLE 1. *A system with a unique solution.* Consider the system

$$\begin{aligned} 2x - 5y + 4z &= -3 \\ x - 2y + z &= 5 \\ x - 4y + 6z &= 10 . \end{aligned}$$

This particular system has a unique solution, $x = 124$, $y = 75$, $z = 31$, which we shall obtain by the Gauss-Jordan elimination process. To save labor we do not bother to copy the letters x, y, z and the equals sign over and over again, but work instead with the *augmented matrix*

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 1 & -2 & 1 & 5 \\ 1 & -4 & 6 & 10 \end{array}\right] \tag{16.24}$$

obtained by adjoining the right-hand members of the system to the coefficient matrix. The three basic types of operations mentioned above are performed on the rows of the augmented matrix and are called *row operations*. At any stage of the process we can put the letters x, y, z back again and insert equals signs along the vertical line to obtain equations. Our ultimate goal is to arrive at the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \end{array}\right] \tag{16.25}$$

after a succession of row operations. The corresponding system of equations is $x = 124$, $y = 75$, $z = 31$, which gives the desired solution.

The first step is to obtain a 1 in the upper left-hand corner of the matrix. We can do this by interchanging the first row of the given matrix (16.24) with either the second or third row. Or, we can multiply the first row by $\frac{1}{2}$. Interchanging the first and second rows, we get

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 2 & -5 & 4 & -3 \\ 1 & -4 & 6 & 10 \end{array}\right].$$

The next step is to make all the remaining entries in the first column equal to zero, leaving the first row intact. To do this we multiply the first row by -2 and add the result to the second row. Then we multiply the first row by -1 and add the result to the third row. After these two operations, we obtain

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & -1 & 2 & -13 \\ 0 & -2 & 5 & 5 \end{array}\right]. \tag{16.26}$$

Now we repeat the process on the smaller matrix $\left[\begin{array}{cc|c} -1 & 2 & -13 \\ -2 & 5 & 5 \end{array}\right]$ which appears adjacent to the two zeros. We can obtain a 1 in *its* upper left-hand corner by multiplying the second row of (16.26) by -1. This gives us the matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & -2 & 13 \\ 0 & -2 & 5 & 5 \end{array}\right].$$

Multiplying the second row by 2 and adding the result to the third, we get

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 1 & 31 \end{array}\right]. \tag{16.27}$$

At this stage, the corresponding system of equations is given by

$$\begin{aligned} x - 2y + z &= 5 \\ y - 2z &= 13 \\ z &= 31 . \end{aligned}$$

These equations can be solved in succession, starting with the third one and working backwards, to give us

$$z = 31, \qquad y = 13 + 2z = 13 + 62 = 75, \qquad x = 5 + 2y - z = 5 + 150 - 31 = 124 .$$

Or, we can continue the Gauss–Jordan process by making all the entries zero above the diagonal elements in the second and third columns. Multiplying the second row of (16.27) by 2 and adding the result to the first row, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 31 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 1 & 31 \end{array}\right].$$

Finally, we multiply the third row by 3 and add the result to the first row, and then multiply the third row by 2 and add the result to the second row to get the matrix in (16.25).

EXAMPLE 2. *A system with more than one solution.* Consider the following system of 3 equations in 5 unknowns:

$$\begin{aligned} 2x - 5y + 4z + u - v &= -3 \\ x - 2y + z - u + v &= 5 \\ x - 4y + 6z + 2u - v &= 10 . \end{aligned} \tag{16.28}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccccc|c} 2 & -5 & 4 & 1 & -1 & -3 \\ 1 & -2 & 1 & -1 & 1 & 5 \\ 1 & -4 & 6 & 2 & -1 & 10 \end{array}\right].$$

The coefficients of x, y, z and the right-hand members are the same as those in Example 1.

If we perform the same row operations used in Example 1, we finally arrive at the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -16 & 19 & 124 \\ 0 & 1 & 0 & -9 & 11 & 75 \\ 0 & 0 & 1 & -3 & 4 & 31 \end{array}\right].$$

The corresponding system of equations can be solved for x, y, and z in terms of u and v, giving us

$$\begin{aligned} x &= 124 + 16u - 19v \\ y &= 75 + 9u - 11v \\ z &= 31 + 3u - 4v . \end{aligned}$$

If we let $u = t_1$ and $v = t_2$, where t_1 and t_2 are arbitrary real numbers, and determine x, y, z by these equations, the vector (x, y, z, u, v) in V_5 given by

$$(x, y, z, u, v) = (124 + 16t_1 - 19t_2 , 75 + 9t_1 - 11t_2 , 31 + 3t_1 - 4t_2 , t_1 , t_2)$$

is a solution. By separating the parts involving t_1 and t_2, we can rewrite this as follows:

$$(x, y, z, u, v) = (124, 75, 31, 0, 0) + t_1(16, 9, 3, 1, 0) + t_2(-19, -11, -4, 0, 1) .$$

This equation gives the general solution of the system. The vector (124, 75, 31, 0, 0) is a particular solution of the nonhomogeneous system (16.28). The two vectors (16, 9, 3, 1, 0) and (−19, −11, −4, 0, 1) are solutions of the corresponding homogeneous system. Since they are independent, they form a basis for the space of all solutions of the homogeneous system.

EXAMPLE 3. *A system with no solution.* Consider the system

$$\begin{aligned} 2x - 5y + 4z &= -3 \\ x - 2y + z &= 5 \\ x - 4y + 5z &= 10 . \end{aligned} \tag{16.29}$$

This system is almost identical to that of Example 1 except that the coefficient of z in the third equation has been changed from 6 to 5. The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 1 & -2 & 1 & 5 \\ 1 & -4 & 5 & 10 \end{array}\right].$$

Applying the same row operations used in Example 1 to transform (16.24) into (16.27), we arrive at the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 0 & 31 \end{array}\right]. \tag{16.30}$$

When the bottom row is expressed as an equation, it states that $0 = 31$. Therefore the original system has no solution since the two systems (16.29) and (16.30) are equivalent.

In each of the foregoing examples, the number of equations did not exceed the number of unknowns. If there are more equations than unknowns, the Gauss–Jordan process is still applicable. For example, suppose we consider the system of Example 1, which has the solution $x = 124, y = 75, z = 31$. If we adjoin a new equation to this system which is also satisfied by the same triple, for example, the equation $2x - 3y + z = 54$, then the elimination process leads to the agumented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

with a row of zeros along the bottom. But if we adjoin a new equation which is not satisfied by the triple (124, 75, 31), for example the equation $x + y + z = 1$, then the elimination process leads to an augmented matrix of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \\ 0 & 0 & 0 & a \end{array}\right],$$

where $a \neq 0$. The last row now gives a contradictory equation $0 = a$ which shows that the system has no solution.

16.19 Inverses of square matrices

Let $A = (a_{ij})$ be a square $n \times n$ matrix. If there is another $n \times n$ matrix B such that $BA = I$, where I is the $n \times n$ identity matrix, then A is called *nonsingular* and B is called a *left inverse* of A.

Choose the usual basis of unit coordinate vectors in V_n and let $T: V_n \to V_n$ be the linear transformation with matrix $m(T) = A$. Then we have the following.

THEOREM 16.20. *The matrix A is nonsingular if and only if T is invertible. If $BA = I$, then $B = m(T^{-1})$.*

Proof. Assume that A is nonsingular and that $BA = I$. We shall prove that $T(x) = O$ implies $x = O$. Given x such that $T(x) = O$, let X be the $n \times 1$ column matrix formed from the components of x. Since $T(x) = O$, the matrix product AX is an $n \times 1$ column matrix consisting of zeros, so $B(AX)$ is also a column matrix of zeros. But $B(AX) = (BA)X = IX = X$, so every component of x is 0. Therefore, T is invertible, and the equation $TT^{-1} = I$ implies that $m(T)m(T^{-1}) = I$ or $Am(T^{-1}) = I$. Multiplying on the left by B, we find $m(T^{-1}) = B$. Conversely, if T is invertible, then $T^{-1}T$ is the identity transformation so $m(T^{-1})m(T)$ is the identity matrix. Therefore A is nonsingular and $m(T^{-1})A = I$.

All the properties of invertible linear transformations have their counterparts for nonsingular matrices. In particular, left inverses (if they exist) are unique, and every left inverse is also a right inverse. In other words, if A is nonsingular and $BA = I$, then $AB = I$. We call B the *inverse* of A and denote it by A^{-1}. The inverse A^{-1} is also nonsingular and *its* inverse is A.

Now we show that the problem of actually determining the entries of the inverse of a nonsingular matrix is equivalent to solving n separate nonhomogeneous linear systems.

Let $A = (a_{ij})$ be nonsingular and let $A^{-1} = (b_{ij})$ be its inverse. The entries of A and A^{-1} are related by the n^2 equations

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \delta_{ij}, \tag{16.31}$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. For each fixed choice of j, we can regard this as a nonhomogeneous system of n linear equations in n unknowns $b_{1j}, b_{2j}, \ldots, b_{nj}$. Since A is nonsingular, each of these systems has a unique solution, the jth column of B. All these systems have the same coefficient-matrix A and differ only in their right members. For example, if A is a 3×3 matrix, there are 9 equations in (16.31) which can be expressed as 3 separate linear systems having the following augmented matrices:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 1 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 1 \end{array}\right].$$

If we apply the Gauss–Jordan process, we arrive at the respective augmented matrices

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_{11} \\ 0 & 1 & 0 & b_{21} \\ 0 & 0 & 1 & b_{31} \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_{12} \\ 0 & 1 & 0 & b_{22} \\ 0 & 0 & 1 & b_{32} \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_{13} \\ 0 & 1 & 0 & b_{23} \\ 0 & 0 & 1 & b_{33} \end{array}\right].$$

In actual practice we exploit the fact that all three systems have the same coefficient-matrix and solve all three systems at once by working with the enlarged matrix

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right].$$

The elimination process then leads to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right].$$

The matrix on the right of the vertical line is the required inverse. The matrix on the left of the line is the 3×3 identity matrix.

It is not necessary to know in advance whether A is nonsingular. If A is *singular* (not nonsingular), we can still apply the Gauss–Jordan method, but somewhere in the process one of the diagonal elements will become zero, and it will not be possible to transform A to the identity matrix.

16.20 Exercises

Apply the Gauss–Jordan elimination process to each of the following systems. If a solution exists, determine the general solution.

1. $x + y + 3z = 5$, $2x - y + 4z = 11$, $-y + z = 3$.
2. $3x + 2y + z = 1$, $5x + 3y + 3z = 2$, $x + y - z = 1$.
3. $3x + 2y + z = 1$, $5x + 3y + 3z = 2$, $7x + 4y + 5z = 3$.
4. $3x + 2y + z = 1$, $5x + 3y + 3z = 2$, $7x + 4y + 5z = 3$, $x + y - z = 0$.
5. $3x - 2y + 5z + u = 1$, $x + y - 3z + 2u = 2$, $6x + y - 4z + 3u = 7$.
6. $x + y - 3z + u = 5$, $2x - y + z - 2u = 2$, $7x + y - 7z + 3u = 3$.
7. $x + y + 2z + 3u + 4v = 0$, $2x + 2y + 7z + 11u + 14v = 0$, $3x + 3y + 6z + 10u + 15v = 0$.
8. $x - 2y + z + 2u = -2$, $2x + 3y - z - 5u = 9$, $4x - y + z - u = 5$, $5x - 3y + 2z + u = 3$.
9. Prove that the system $x + y + 2z = 2$, $2x - y + 3z = 2$, $5x - y + az = 6$, has a unique solution if $a \neq 8$. Find all solutions when $a = 8$.
10. (a) Determine all solutions of the system

$$\begin{aligned} 5x + 2y - 6z + 2u &= -1 \\ x - y + z - u &= -2. \end{aligned}$$

(b) Determine all solutions of the system

$$\begin{aligned} 5x + 2y - 6z + 2u &= -1 \\ x - y + z - u &= -2 \\ x + y + z &= 6. \end{aligned}$$

11. This exercise tells how to determine all nonsingular 2×2 matrices. Prove that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad - bc)I.$$

Deduce that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular if and only if $ad - bc \neq 0$, in which case its inverse is

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$