The following is a proof of the 2nd order Taylor's formula for scalar functions of several variables, under the assumption that f is merely C^2 (the proof in the book assumes C^3).

Theorem. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 and fix $\mathbf{a} \in \mathbb{R}^n$. Then, for all $\mathbf{h} \in \mathbb{R}^n$, we can write*

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + Df(\mathbf{a})\,\mathbf{h} + \frac{1}{2}\mathbf{h}^T\, Hf(\mathbf{a})\,\mathbf{h} + E_2(\mathbf{h}). \tag{1}$$

Here $Hf(\mathbf{a}) = [\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})]$ is the Hessian matrix of f at $\mathbf{a}$, and $E_2(\mathbf{h})$ is an error term which satisfies

$$\frac{E_2(\mathbf{h})}{\|\mathbf{h}\|^2} \to 0 \qquad \text{as } \mathbf{h} \to \mathbf{0}.$$

The idea is to apply the 2nd order Taylor's formula to a suitable function of a single variable. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = f(\mathbf{a}+t\mathbf{h}). \tag{2}$$

By the chain rule, g is differentiable and

$$g'(t) = Df(\mathbf{a}+t\mathbf{h})\,\mathbf{h}. \tag{3}$$

To examine the possibility of taking another derivative, it will be convenient to write g' in terms of the components of the vectors involved. Assuming $\mathbf{h} = (h_1, h_2, \ldots, h_n)$, we can write (3) as

$$g'(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{a}+t\mathbf{h})\,h_j.$$

By the assumption, f is C^2 so each partial derivative $\partial f/\partial x_j$ is differentiable on $\mathbb{R}^n$. Applying the chain rule once more, it follows that g'' exists and

$$g''(t) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}+t\mathbf{h})\,h_i\right)h_j = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}+t\mathbf{h})\,h_i h_j$$

which can be put in the matrix form

$$g''(t) = \mathbf{h}^T\, Hf(\mathbf{a}+t\mathbf{h})\,\mathbf{h}. \tag{4}$$

Since f is C^2, the entries of the Hessian Hf are continuous on $\mathbb{R}^n$. Hence the entries of $Hf(\mathbf{a}+t\mathbf{h})$, being compositions of continuous functions, are continuous in t. It follows that g'' is continuous.

Thus, we can apply the 2nd order Taylor's formula of one-variable calculus to g on the interval $0 \le t \le 1$:

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(c) \qquad \text{for some } 0 < c < 1.$$

Substituting various terms from (2), (3) and (4), we obtain

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + Df(\mathbf{a})\,\mathbf{h} + \frac{1}{2}\,\mathbf{h}^T\, Hf(\mathbf{a}+c\mathbf{h})\,\mathbf{h}$$

If we define

$$E_2(\mathbf{h}) = \frac{1}{2}\,\mathbf{h}^T\, Hf(\mathbf{a}+c\mathbf{h})\,\mathbf{h} - \frac{1}{2}\,\mathbf{h}^T\, Hf(\mathbf{a})\,\mathbf{h},$$

it follows that (1) holds. It remains to show that $E_2(\mathbf{h})/\|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$.

To this end, note that

$$E_2(\mathbf{h}) = \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}+c\mathbf{h}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})\right) h_i h_j$$

so by the triangle inequality

$$|E_2(\mathbf{h})| \le \frac{1}{2}\sum_{i,j=1}^{n}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}+c\mathbf{h}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})\right| |h_i|\;|h_j|$$

Each product $|h_i|\;|h_j|$ is at most h_i^2 or h_j^2. In either case, $|h_i|\;|h_j| \le h_1^2 + \cdots + h_n^2 = \|\mathbf{h}\|^2$. It follows that

$$|E_2(\mathbf{h})| \le \frac{1}{2}\sum_{i,j=1}^{n}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}+c\mathbf{h}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})\right| \|\mathbf{h}\|^2,$$

or, dividing by $\|\mathbf{h}\|^2$,

$$\frac{E_2(\mathbf{h})}{\|\mathbf{h}\|^2} \le \frac{1}{2}\sum_{i,j=1}^{n}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}+c\mathbf{h}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})\right|.$$

As $\mathbf{h} \to \mathbf{0}$, $\mathbf{a}+c\mathbf{h} \to \mathbf{a}$ and from continuity of the second partial derivatives $\partial^2 f/\partial x_i \partial x_j$ it follows that each term on the right tends to zero. Hence $E_2(\mathbf{h})/\|\mathbf{h}\|^2$ must tend to zero as well.

Challenge. Can you push this proof further and obtain the k-th order Taylor's formula in several variables? This would require finding formulas for higher derivatives of g. As a test case, suppose $n = 2$ and try to obtain a 3rd order formula by finding $g'''(t)$, writing

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) + \frac{1}{6}g'''(c) \qquad \text{for some } 0 < c < 1,$$

and substituting in terms of f.