The Cauchy-Schwarz inequality. *For any pair of vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*, we have*

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Here is a very simple proof of this inequality. If $\mathbf{u} = \mathbf{0}$, both sides of the inequality are zero and there is nothing to prove. So let us assume $\mathbf{u} \neq \mathbf{0}$. Consider the function

$$f(t) = \|t\mathbf{u} + \mathbf{v}\|^2$$

of the real variable t, which satisfies $f(t) \geq 0$ for all t. It is easy to see that $f(t)$ is a quadratic polynomial of the form $at^2 + bt + c$ with $a > 0$. In fact,

$$\begin{aligned} f(t) &= (t\mathbf{u} + \mathbf{v}) \cdot (t\mathbf{u} + \mathbf{v}) \\ &= (t\mathbf{u}) \cdot (t\mathbf{u}) + (t\mathbf{u}) \cdot \mathbf{v} + \mathbf{v} \cdot (t\mathbf{u}) + \mathbf{v} \cdot \mathbf{v} \\ &= \underbrace{\|\mathbf{u}\|^2}_{a} t^2 + \underbrace{2(\mathbf{u} \cdot \mathbf{v})}_{b} t + \underbrace{\|\mathbf{v}\|^2}_{c}. \end{aligned}$$

Now the discriminant $b^2 - 4ac$ of this quadratic polynomial cannot be positive since in that case f would have two distinct roots and any t between those roots would satisfy $f(t) < 0$, contrary to the fact that $f(t) \geq 0$ for all values of t. Thus,

$$b^2 - 4ac \leq 0 \Longrightarrow 4(\mathbf{u} \cdot \mathbf{v})^2 - 4\|\mathbf{u}\|^2\|\mathbf{v}\|^2 \leq 0 \Longrightarrow (\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2.$$

Taking the square roots of both sides, we obtain the Cauchy-Schwarz inequality.