

Math 231 First Midterm Solutions

Problem 1. Mark each of the following statements as true (T) or false (F). Briefly give a reason.

- Every system of 3 linear equations in 4 unknowns is consistent.

False. For example, the system

$$\begin{cases} x + y + z + w = -1 \\ x + y + z + w = 0 \\ x - y + z - w = 1 \end{cases}$$

has no solutions.

- If A is a 3×1 matrix such that $A^T A = 0$, then $A = 0$.

True. Suppose $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so that $A^T A$ is the 1×1 matrix

$$[a \ b \ c] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a^2 + b^2 + c^2].$$

Since $A^T A = 0$, we have $a^2 + b^2 + c^2 = 0$, which implies $a = b = c = 0$, so A is the 3×1 zero matrix.

- Every symmetric triangular matrix is diagonal.

True. For example, if A is upper triangular then all the entries below its main diagonal are zero. Moreover, A being symmetric implies that all the entries above its main diagonal are also zero. Hence A has to be diagonal.

- There is a square matrix A such that $\det(A^2) = -5$.

False. There is no such matrix because

$$-5 = \det(A^2) = \det(AA) = \det(A) \det(A) = (\det(A))^2.$$

and the square of no real number can be -5 .

Problem 2. The augmented matrix of a system of linear equations has been reduced to the row-echelon form

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 2 & \vdots & 3 \\ 0 & 1 & 0 & 1 & 2 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Call the unknowns x_1, x_2, x_3 , etc.

- (i) How many unknowns are there? Which are the “leading” variables? Which are the “free” variables?

There are six unknowns: x_1, x_2, x_5 are leading and x_3, x_4, x_6 are free.

- (ii) Find all solutions of this system.

We assign arbitrary values to the free variables and express the leading variables in terms of them. Let $x_3 = r, x_4 = s, x_6 = t$. Then we have

$$\begin{cases} x_1 + 2x_3 + 3x_5 + 2x_6 = 3 \\ x_2 + x_4 + 2x_5 + x_6 = 1 \\ x_5 = 2 \end{cases}$$

which gives

$$\begin{cases} x_5 = 2 \\ x_2 = 1 - x_4 - 2x_5 - x_6 = 1 - s - 4 - t = -3 - s - t \\ x_1 = 3 - 2x_3 - 3x_5 - 2x_6 = 3 - 2r - 6 - 2t = -3 - 2r - 2t \end{cases}$$

So the general solution of this system is

$$\begin{cases} x_1 = -3 - 2r - 2t \\ x_2 = -3 - s - t \\ x_3 = r \\ x_4 = s \\ x_5 = 2 \\ x_6 = t \end{cases}$$

where r, s, t are arbitrary real numbers.

Problem 3. Consider the linear system

$$\begin{cases} 3y + z = 3 \\ x - 8y - 2z = -2 \\ x + 2y + z = 0 \end{cases}$$

- (i) Write this system as $\mathbf{Ax} = \mathbf{b}$ by introducing the matrices A , \mathbf{x} and \mathbf{b} .

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & -8 & -2 \\ 1 & 2 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

- (ii) Find the inverse A^{-1} by applying elementary row operations on the identity matrix.

$$\begin{aligned}
& \left[\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & -8 & -2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -8 & -2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \\
& \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & -8 & -2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 10 & 3 & 0 & -1 & 1 \end{array} \right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & -8 & -2 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 10 & 3 & 0 & -1 & 1 \end{array} \right] \\
& \xrightarrow{-10R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -8 & -2 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & -\frac{10}{3} & -1 & 1 \end{array} \right] \xrightarrow{-3R_3} \left[\begin{array}{ccc|ccc} 1 & -8 & -2 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 10 & 3 & -3 \end{array} \right] \\
& \xrightarrow{-\frac{1}{3}R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -8 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & -1 & 1 \\ 0 & 0 & 1 & 10 & 3 & -3 \end{array} \right] \xrightarrow{2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & -8 & 0 & 20 & 7 & -6 \\ 0 & 1 & 0 & -3 & -1 & 1 \\ 0 & 0 & 1 & 10 & 3 & -3 \end{array} \right] \\
& \xrightarrow{8R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -1 & 2 \\ 0 & 1 & 0 & -3 & -1 & 1 \\ 0 & 0 & 1 & 10 & 3 & -3 \end{array} \right] \Rightarrow A^{-1} = \begin{bmatrix} -4 & -1 & 2 \\ -3 & -1 & 1 \\ 10 & 3 & -3 \end{bmatrix}
\end{aligned}$$

(iii) Using the matrix A^{-1} you found in (ii), find the solution \mathbf{x} of the system.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -4 & -1 & 2 \\ -3 & -1 & 1 \\ 10 & 3 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -12 + 2 + 0 \\ -9 + 2 + 0 \\ 30 - 6 - 0 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \\ 24 \end{bmatrix}.$$

Problem 4. For what values of x is the following matrix invertible?

$$\begin{bmatrix} x-1 & 3 & -1 \\ 3 & -6 & x \\ 7 & 0 & x-2 \end{bmatrix}$$

A square matrix is invertible if and only if its determinant is non-zero. By cofactor expansion along the third row, the determinant of the above matrix is seen to be

$$(7)(3x-6) + (x-2)[(x-1)(-6) - 9] = 21x - 42 + (x-2)(-6x-3) = -6x^2 + 30x - 36.$$

Thus the determinant is zero if and only if $-6x^2 + 30x - 36 = 0$ which has the solutions $x = 2$ and $x = 3$. It follows that the matrix is invertible for all values of x except $x = 2$ and $x = 3$.

Bonus Problem. Suppose A is a 3×3 matrix with the property that

$$A^T = -A.$$

Find $\text{tr}(A)$ and $\det(A)$.

Solution 1. Using the fact that $\text{tr}(A^T) = \text{tr}(A)$, we have

$$\begin{aligned}A^T = -A &\implies \text{tr}(A^T) = \text{tr}(-A) \\ &\implies \text{tr}(A) = -\text{tr}(A) \\ &\implies \text{tr}(A) = 0.\end{aligned}$$

Similarly, using the fact that $\det(A^T) = \det(A)$, we have

$$\begin{aligned}A^T = -A &\implies \det(A^T) = \det(-A) \\ &\implies \det(A) = (-1)^3 \det(A) \\ &\implies \det(A) = -\det(A) \\ &\implies \det(A) = 0.\end{aligned}$$

Solution 2. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

so the condition $A^T = -A$ means

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}.$$

This gives

$$a = e = i = 0 \quad d = -b \quad g = -c \quad h = -f.$$

Thus A has the form

$$A = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix}.$$

It follows that $\text{tr}(A) = 0$. Moreover, cofactor expansion along the first row gives

$$\det(A) = (-b)(cf) + (c)(bf) = 0.$$