

Math 231 Second Midterm Solutions

Problem 1. Mark each of the following statements as true or false. Give a brief reason.

- If \mathbf{u} is orthogonal to \mathbf{v} , then $2\mathbf{u}$ is orthogonal to $-\mathbf{v}$.

True. If \mathbf{u} is orthogonal to \mathbf{v} , then $\mathbf{u} \cdot \mathbf{v} = 0$. But then

$$2\mathbf{u} \cdot -\mathbf{v} = (2)(-1)\mathbf{u} \cdot \mathbf{v} = (-2)(0) = 0$$

which means $2\mathbf{u}$ is orthogonal to $-\mathbf{v}$.

- If the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span a vector space V , then $\dim V = 3$.

False, because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ may be linearly dependent. For example, the vectors $\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = (0, 1), \mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 = (1, 1)$ span \mathbb{R}^2 , but $\dim \mathbb{R}^2 = 2$.

- It is possible to find 7 linearly independent vectors in \mathbb{R}^6 .

False, because $\dim \mathbb{R}^6 = 6$ so any collection of more than 6 vectors in \mathbb{R}^6 must be linearly dependent.

- There is a 3×5 matrix A whose rank is 4.

False, because if A is 3×5 , then $\text{Col}(A)$ is a subspace of \mathbb{R}^3 so $\text{rank}(A) = \dim \text{Col}(A) \leq 3$. Alternatively, $\text{Row}(A)$ is spanned by 3 vectors in \mathbb{R}^5 , so $\text{rank}(A) = \dim \text{Row}(A) \leq 3$.

Problem 2. Consider the vectors

$$\mathbf{u} = (-1, 0, 1) \quad \text{and} \quad \mathbf{v} = (0, 0, 2)$$

in \mathbb{R}^3 . Find all scalars a, b such that the vector $a\mathbf{u} + b\mathbf{v}$ is orthogonal to \mathbf{v} and has norm 1.

Write

$$\mathbf{w} = a\mathbf{u} + b\mathbf{v} = (-a, 0, a) + (0, 0, 2b) = (-a, 0, a + 2b).$$

We want \mathbf{w} to be orthogonal to \mathbf{v} , so

$$\mathbf{w} \cdot \mathbf{v} = (-a, 0, a + 2b) \cdot (0, 0, 2) = 2(a + 2b) = 0 \implies a + 2b = 0.$$

We also want \mathbf{w} to have norm 1, so

$$\begin{aligned} \|\mathbf{w}\| = 1 &\implies \sqrt{(-a)^2 + 0^2 + (a + 2b)^2} = 1 \\ &\implies \sqrt{a^2} = 1 \quad (\text{because } a + 2b = 0) \\ &\implies |a| = 1 \\ &\implies a = 1 \text{ or } a = -1. \end{aligned}$$

Using $a + 2b = 0$, we obtain two sets of solutions:

$$a = 1, b = -\frac{1}{2} \quad \text{or} \quad a = -1, b = \frac{1}{2}.$$

Problem 3. Let W be the set of all 2×2 matrices A such that $\text{tr}(A) = 0$. In other words, W consists of all matrices of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

where a, b, c are arbitrary real numbers.

(i) Verify that W is a subspace of $\mathcal{M}_{2,2}$.

We must verify that W is closed under addition and scalar multiplication:

- Suppose $A, B \in W$ so that $\text{tr}(A) = \text{tr}(B) = 0$. Then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) = 0 + 0 = 0$, so $A + B \in W$.
- Suppose λ is a scalar and $A \in W$ so that $\text{tr}(A) = 0$. Then $\text{tr}(\lambda A) = \lambda \text{tr}(A) = \lambda \cdot 0 = 0$, so $\lambda A \in W$.

(ii) Find a basis and dimension of W .

Every element of W can be written as

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It follows that the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

span W . Moreover, A_1, A_2, A_3 are linearly independent because

$$aA_1 + bA_2 + cA_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies a = b = c = 0.$$

Thus $\{A_1, A_2, A_3\}$ forms a basis for W . In particular, $\dim W = 3$.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 3 & 1 & 5 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 8 \end{bmatrix}$$

(i) For what vectors $\mathbf{b} \in \mathbb{R}^4$ is the system $A\mathbf{x} = \mathbf{b}$ consistent?

We form the augmented matrix $[A : \mathbf{b}]$ and reduce it to row-echelon form:

$$\begin{aligned} & \begin{bmatrix} 0 & 0 & 3 & 1 & 5 & : & a \\ 1 & 3 & 1 & 2 & 1 & : & b \\ 3 & 9 & 4 & 5 & 2 & : & c \\ 4 & 12 & 8 & 8 & 8 & : & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 & : & b \\ 0 & 0 & 3 & 1 & 5 & : & a \\ 3 & 9 & 4 & 5 & 2 & : & c \\ 4 & 12 & 8 & 8 & 8 & : & d \end{bmatrix} \\ & \xrightarrow{-3R_1 + R_3, -4R_1 + R_4} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 & : & b \\ 0 & 0 & 3 & 1 & 5 & : & a \\ 0 & 0 & 1 & -1 & -1 & : & -3b + c \\ 0 & 0 & 4 & 0 & 4 & : & -4b + d \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 & : & b \\ 0 & 0 & 1 & -1 & -1 & : & -3b + c \\ 0 & 0 & 3 & 1 & 5 & : & a \\ 0 & 0 & 4 & 0 & 4 & : & -4b + d \end{bmatrix} \end{aligned}$$

$$\begin{array}{c}
-3R_2+R_3, -4R_2+R_4 \\
\longrightarrow
\end{array}
\begin{bmatrix}
1 & 3 & 1 & 2 & 1 & \vdots & b \\
0 & 0 & 1 & -1 & -1 & \vdots & -3b+c \\
0 & 0 & 0 & 4 & 8 & \vdots & a+9b-3c \\
0 & 0 & 0 & 4 & 8 & \vdots & 8b-4c+d
\end{bmatrix}
\begin{array}{c}
-R_3+R_4 \\
\longrightarrow
\end{array}
\begin{bmatrix}
1 & 3 & 1 & 2 & 1 & \vdots & b \\
0 & 0 & 1 & -1 & -1 & \vdots & -3b+c \\
0 & 0 & 0 & 4 & 8 & \vdots & a+9b-3c \\
0 & 0 & 0 & 0 & 0 & \vdots & -a-b-c+d
\end{bmatrix}$$

$$\begin{array}{c}
\frac{1}{4}R_3 \\
\longrightarrow
\end{array}
\begin{bmatrix}
1 & 3 & 1 & 2 & 1 & \vdots & b \\
0 & 0 & 1 & -1 & -1 & \vdots & -3b+c \\
0 & 0 & 0 & 1 & 2 & \vdots & \frac{1}{4}(a+9b-3c) \\
0 & 0 & 0 & 0 & 0 & \vdots & -a-b-c+d
\end{bmatrix}$$

The consistency condition is therefore $-a - b - c + d = 0$. In other words,

$$\mathbf{Ax} = \mathbf{b} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ is consistent } \iff a + b + c = d.$$

(ii) Find bases for $\text{Row}(A)$, $\text{Col}(A)$ and $\text{Null}(A)$.

By part (i), a row-echelon form of A is

$$R = \begin{bmatrix}
1 & 3 & 1 & 2 & 1 \\
0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Thus the vectors

$$\mathbf{u}_1 = [1 \ 3 \ 1 \ 2 \ 1] \quad \mathbf{u}_2 = [0 \ 0 \ 1 \ -1 \ -1] \quad \mathbf{u}_3 = [0 \ 0 \ 0 \ 1 \ 2]$$

form a basis for $\text{Row}(A)$. The leading 1's of R occur along the first, third and fourth columns, so the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{c}_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 8 \end{bmatrix} \quad \mathbf{c}_4 = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 8 \end{bmatrix}$$

form a basis for $\text{Col}(A)$. To find a basis for $\text{Null}(A)$, we must solve the homogeneous system $\mathbf{Ax} = \mathbf{0}$, or equivalently $R\mathbf{x} = \mathbf{0}$:

$$\begin{cases}
x_1 + 3x_2 + x_3 + 2x_4 + x_5 = 0 \\
x_3 - x_4 - x_5 = 0 \\
x_4 + 2x_5 = 0
\end{cases}$$

The leading variables are x_1, x_3, x_4 and the free variables are x_2, x_5 . Hence the above system can be solved by assigning arbitrary values to x_2, x_5 and expressing x_1, x_3, x_4

in terms of them:

$$\begin{cases} x_1 = -3s + 4t \\ x_2 = s \\ x_3 = -t \\ x_4 = -2t \\ x_5 = t \end{cases} \quad s, t \in \mathbb{R}$$

Thus

$$\mathbf{x} \in \text{Null}(A) \iff \mathbf{x} = \begin{bmatrix} -3s + 4t \\ s \\ -t \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \quad s, t \in \mathbb{R}$$

It follows that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

form a basis for $\text{Null}(A)$.

(iii) Find $\text{rank}(A)$ and $\text{nullity}(A)$ and verify that $\text{rank}(A) + \text{nullity}(A) = 5$.

We have

$$\text{rank}(A) = \dim \text{Row}(A) = \dim \text{Col}(A) = 3$$

and

$$\text{nullity}(A) = \dim \text{Null}(A) = 2.$$

Bonus Problem. Let A be an $n \times n$ matrix such that $\det(A) = 0$. Show that the rows of A , viewed as vectors in \mathbb{R}^n , must be linearly dependent.

Reduce A to an $n \times n$ row-echelon matrix R . Since $\det(A) = 0$, we have $\det(R) = 0$, so R must have an all-zero row. Hence $\text{rank}(A)$ (which is the number of non-zero rows of R) is less than n . This shows the rows of A must be dependent since otherwise $\text{rank}(A)$ would be n .