Lemma. *Let S be a subspace of $\mathbb{R}^n$. If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent vectors in S and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ span S, then $k \le m$.*

In other words, linearly independent sets cannot have more vectors than spanning sets.

Proof. Since $S = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, each $\mathbf{u}_i$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$:

$$\begin{aligned} \mathbf{u}_1 &= a_{11}\,\mathbf{v}_1 + a_{21}\,\mathbf{v}_2 + \cdots + a_{m1}\,\mathbf{v}_m \\ \mathbf{u}_2 &= a_{12}\,\mathbf{v}_1 + a_{22}\,\mathbf{v}_2 + \cdots + a_{m2}\,\mathbf{v}_m \\ &\ \ \vdots \qquad \vdots \qquad \vdots \\ \mathbf{u}_k &= a_{1k}\,\mathbf{v}_1 + a_{2k}\,\mathbf{v}_2 + \cdots + a_{mk}\,\mathbf{v}_m \end{aligned}$$

Let us form the linear combination $c_1\,\mathbf{u}_1 + c_2\,\mathbf{u}_2 + \cdots + c_k\,\mathbf{u}_k$ by multiplying the first equation above by c_1, the second by c_2, ..., the k-th by c_k, and adding the results vertically:

$$\begin{aligned} c_1\,\mathbf{u}_1 + c_2\,\mathbf{u}_2 + \cdots + c_k\,\mathbf{u}_k = &\ \big(a_{11}\,c_1 + a_{12}\,c_2 + \cdots + a_{1k}\,c_k\big)\,\mathbf{v}_1 \\ &+ \big(a_{21}\,c_1 + a_{22}\,c_2 + \cdots + a_{2k}\,c_k\big)\,\mathbf{v}_2 \\ &+ \cdots \\ &+ \big(a_{m1}\,c_1 + a_{m2}\,c_2 + \cdots + a_{mk}\,c_k\big)\,\mathbf{v}_m \end{aligned}$$

Setting the coefficients of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in the above expression equal to zero, we obtain the following homogeneous system of m linear equations in k unknowns $c_1, \ldots, c_k$:

$$\begin{aligned} a_{11}\,c_1 + a_{12}\,c_2 + \cdots + a_{1k}\,c_k &= 0 \\ a_{21}\,c_1 + a_{22}\,c_2 + \cdots + a_{2k}\,c_k &= 0 \\ \vdots \qquad \vdots \qquad \vdots \ \ & \\ a_{m1}\,c_1 + a_{m2}\,c_2 + \cdots + a_{mk}\,c_k &= 0 \end{aligned}$$

If $m < k$, there would be fewer equations than unknowns, so the system would have a non-trivial solution for $c_1, \ldots, c_k$. But for this solution we would have $c_1\,\mathbf{u}_1 + c_2\,\mathbf{u}_2 + \cdots + c_k\,\mathbf{u}_k = 0$ and this would contradict the assumption that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent. Thus $m < k$ cannot happen and we must have $k \le m$. $\square$