Math 231 Midterm 1 Review Sheet, 2/28/2024

Here is a list of the topics that could be on the first midterm exam. Make sure you study each item carefully.

- Vectors in $\mathbb{R}^n$ and their components; vector addition and scalar multiplication; dot product and its basic properties; norm and distance; the triangle and Cauchy-Schwarz inequalities; the angle between two vectors; orthogonal vectors; orthogonal projection of one vector onto another.

- Solving systems of linear equations by elimination, i.e., by reducing the associated augmented matrix to a row echelon form; leading and free variables; every consistent (in particular, homogeneous) system with fewer equations than unknowns has infinitely many solutions.

- Linear combinations of vectors; the span of a set of vectors; geometric view of the span in $\mathbb{R}^2$ and $\mathbb{R}^3$; linear dependence versus independence.

- Place the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ as the *columns* of an $n \times k$ matrix A. Then

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{b} \iff A \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \mathbf{b}.$$

  Thus, we have the following two tests:

  Span Test:

$$\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \iff A\mathbf{x} = \mathbf{b} \text{ has a solution.}$$

  Dependence Test:

$$\mathbf{v}_1, \ldots, \mathbf{v}_k \text{ dependent} \iff A\mathbf{x} = \mathbf{0} \text{ has infinitely many solutions.}$$

  As a result, every collection of more than n vectors in $\mathbb{R}^n$ must be dependent.

- Addition and multiplication of matrices; the transpose of a matrix; symmetric matrices.

- The inverse of a matrix; the formula for the inverse of a 2×2 matrix; if A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

- Elementary matrices; if an elementary row operation applied to I_n gives the elementary matrix E, then the same operation applied to any $n \times n$ matrix A gives the matrix EA.

- Finding the inverse of a matrix by transforming $[A|I]$ to $[I|A^{-1}]$ using elementary row operations.

Practice Problems.

1. Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

in $\mathbb{R}^4$. Find the following quantities:

(i) $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{u}+\mathbf{v}\|$

(ii) The unit vector in the direction of $\mathbf{u} - \mathbf{v}$.

(iii) The angle between $\mathbf{u}$ and $\mathbf{v}$.

(iv) The scalar t such that $\mathbf{u}$ is orthogonal to $\mathbf{u} + t\mathbf{v}$.

(v) $\mathbf{proj_u}(\mathbf{v})$, the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$.

Verify that the triangle and Cauchy-Schwarz inequalities hold for $\mathbf{u}$ and $\mathbf{v}$.

2. Find the general solution of the linear system

$$\begin{cases} x + 3y + 2w = 2 \\ 2x + y + 5z + 4w = -16 \\ 2x + 3y + 3z = 4 \\ 3x + 11y - 2z + 11w = -1 \end{cases}$$

3. Let $A = \begin{bmatrix} 1 & -x \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ 5 & 2 \end{bmatrix}$. Find x such that the product $A^{-1}B^T$ is a symmetric matrix.

4. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 5 \\ 0 & 3 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 4 & 5 & 11 \\ 2 & 1 & 5 \\ 1 & -1 & 0 \end{bmatrix}.$$

(i) Explain how B can be obtained by applying two elementary row operations on A.

(ii) Find elementary matrices E_1 and E_2 such that $B = E_2E_1A$.

5. Consider the linear system

$$\begin{cases} 2x + 4y + z = 1 \\ x + 2y = 2 \\ 3x + 5y + 4z = 0 \end{cases}$$

(i) Write this system as $A\mathbf{x} = \mathbf{b}$ by introducing the matrices A, $\mathbf{x}$ and $\mathbf{b}$.

(ii) Compute the inverse A^{-1} by applying elementary row operations and use it to find the solution $\mathbf{x} = A^{-1}\mathbf{b}$.

6. True or false?

- There is a system of 3 linear equations in 4 unknowns with exactly one solution.
- A 9×7 matrix in row echelon form can have up to 9 leading 1's.
- It is possible to find 4 linearly independent vectors in $\mathbb{R}^3$.
- If A is a 3×1 matrix such that $A^T A = 0$, then $A = 0$.