

Math 231 Practice Test 1 Solutions

Problem 1. (i) Set

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix}.$$

(ii) We form the augmented matrix $[A : \mathbf{b}]$ and reduce it to row-echelon form:

$$\begin{aligned} [A : \mathbf{b}] &= \begin{bmatrix} 2 & 4 & 6 & \vdots & 18 \\ 4 & 5 & 6 & \vdots & 24 \\ 3 & 1 & -2 & \vdots & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 4 & 5 & 6 & \vdots & 24 \\ 3 & 1 & -2 & \vdots & 4 \end{bmatrix} \xrightarrow{-4R_1+R_2, -3R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 0 & -3 & -6 & \vdots & -12 \\ 0 & -5 & -11 & \vdots & -23 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & -5 & -11 & \vdots & -23 \end{bmatrix} \xrightarrow{5R_2+R_3} \begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & -1 & \vdots & -3 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \\ &\xrightarrow{-2R_3+R_2, -3R_3+R_1} \begin{bmatrix} 1 & 2 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & -2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & \vdots & 4 \\ 0 & 1 & 0 & \vdots & -2 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix}. \end{aligned}$$

It follows that $x = 4, y = -2, z = 3$.

(iii) We can find A^{-1} by applying the *same* sequence of elementary row operations on the identity matrix:

$$\begin{aligned} [A : I] &= \begin{bmatrix} 2 & 4 & 6 & \vdots & 1 & 0 & 0 \\ 4 & 5 & 6 & \vdots & 0 & 1 & 0 \\ 3 & 1 & -2 & \vdots & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & \vdots & \frac{1}{2} & 0 & 0 \\ 4 & 5 & 6 & \vdots & 0 & 1 & 0 \\ 3 & 1 & -2 & \vdots & 0 & 0 & 1 \end{bmatrix} \\ &\xrightarrow{-4R_1+R_2, -3R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & -3 & -6 & \vdots & -2 & 1 & 0 \\ 0 & -5 & -11 & \vdots & \frac{-3}{2} & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 3 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & \vdots & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & -5 & -11 & \vdots & \frac{-3}{2} & 0 & 1 \end{bmatrix} \\ &\xrightarrow{5R_2+R_3} \begin{bmatrix} 1 & 2 & 3 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & \vdots & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & -1 & \vdots & \frac{11}{6} & \frac{-5}{3} & 1 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 2 & 3 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & \vdots & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 1 & \vdots & \frac{-11}{6} & \frac{5}{3} & -1 \end{bmatrix} \end{aligned}$$

$$-2R_3+R_2, -3R_3+R_1 \rightarrow \begin{bmatrix} 1 & 2 & 0 & \vdots & 6 & -5 & 3 \\ 0 & 1 & 0 & \vdots & \frac{13}{3} & \frac{-11}{3} & 2 \\ 0 & 0 & 1 & \vdots & \frac{-11}{6} & \frac{5}{3} & -1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{-8}{3} & \frac{7}{3} & -1 \\ 0 & 1 & 0 & \vdots & \frac{13}{3} & \frac{-11}{3} & 2 \\ 0 & 0 & 1 & \vdots & \frac{-11}{6} & \frac{5}{3} & -1 \end{bmatrix} = [I \vdots A^{-1}].$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{-8}{3} & \frac{7}{3} & -1 \\ \frac{13}{3} & \frac{-11}{3} & 2 \\ \frac{-11}{6} & \frac{5}{3} & -1 \end{bmatrix}.$$

(iv)

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{-8}{3} & \frac{7}{3} & -1 \\ \frac{13}{3} & \frac{-11}{3} & 2 \\ \frac{-11}{6} & \frac{5}{3} & -1 \end{bmatrix} \begin{bmatrix} 18 \\ 24 \\ 4 \end{bmatrix} = \begin{bmatrix} (-\frac{8}{3})(18) + (\frac{7}{3})(24) + (-1)(4) \\ (\frac{13}{3})(18) + (-\frac{11}{3})(24) + (2)(4) \\ (-\frac{11}{6})(18) + (\frac{5}{3})(24) + (-1)(4) \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

which is the same answer we found in (i).

Problem 2. Write the system in matrix form and reduce to row-echelon form:

$$\begin{bmatrix} 1 & 3 & 0 & 2 & \vdots & 2 \\ 2 & 1 & 5 & 4 & \vdots & -16 \\ 2 & 3 & 3 & 0 & \vdots & 4 \\ 3 & 11 & -2 & 11 & \vdots & -1 \end{bmatrix} \xrightarrow{-2R_1+R_2, -2R_1+R_3, -3R_1+R_4} \begin{bmatrix} 1 & 3 & 0 & 2 & \vdots & 2 \\ 0 & -5 & 5 & 0 & \vdots & -20 \\ 0 & -3 & 3 & -4 & \vdots & 0 \\ 0 & 2 & -2 & 5 & \vdots & -7 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 3 & 0 & 2 & \vdots & 2 \\ 0 & 1 & -1 & 0 & \vdots & 4 \\ 0 & -3 & 3 & -4 & \vdots & 0 \\ 0 & 2 & -2 & 5 & \vdots & -7 \end{bmatrix} \xrightarrow{3R_2+R_3, -2R_2+R_4} \begin{bmatrix} 1 & 3 & 0 & 2 & \vdots & 2 \\ 0 & 1 & -1 & 0 & \vdots & 4 \\ 0 & 0 & 0 & -4 & \vdots & 12 \\ 0 & 0 & 0 & 5 & \vdots & -15 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{4}R_3} \begin{bmatrix} 1 & 3 & 0 & 2 & \vdots & 2 \\ 0 & 1 & -1 & 0 & \vdots & 4 \\ 0 & 0 & 0 & 1 & \vdots & -3 \\ 0 & 0 & 0 & 5 & \vdots & -15 \end{bmatrix} \xrightarrow{-5R_3+R_4} \begin{bmatrix} 1 & 3 & 0 & 2 & \vdots & 2 \\ 0 & 1 & -1 & 0 & \vdots & 4 \\ 0 & 0 & 0 & 1 & \vdots & -3 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

So the reduced system is

$$\begin{cases} x + 3y + 2w = 2 \\ y - z = 4 \\ w = -3 \end{cases}$$

The solutions of this system are obtained by assigning an arbitrary value t to the “free” variable z and evaluating the “leading” variables x, y, w in terms of it:

$$z = t \quad w = -3 \quad y = z + 4 = t + 4 \quad x = 2 - 3y - 2w = -3t - 4.$$

So the solutions are given by

$$\begin{cases} x &= -3t - 4 \\ y &= t + 4 \\ z &= t \\ w &= -3 \end{cases}$$

where t is an arbitrary real number.

Problem 3. We have

$$B^{-1} = \frac{1}{(7)(2) - (3)(5)} \begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}$$

so

$$AB^{-1} = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix} = \begin{bmatrix} 13 & -30 \\ 13 & -31 \end{bmatrix}.$$

It follows that

$$\text{tr}(AB^{-1}) = 13 - 31 = -18.$$

Problem 4. (i) Simply note that

$$A \xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 5 \\ 4 & 5 & 11 \end{bmatrix} \xrightarrow{R_1 \rightleftharpoons R_3} B$$

(ii) The first row operation $2R_2 + R_3$ corresponds to multiplication from left by

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

The second operation $R_1 \rightleftharpoons R_3$ corresponds to multiplication from left by

$$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $B = E_2 E_1 A$:

$$\begin{bmatrix} 4 & 5 & 11 \\ 2 & 1 & 5 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 5 \\ 0 & 3 & 1 \end{bmatrix}.$$

Problem 5. Form the augmented matrix and reduce it to row-echelon form:

$$\begin{aligned} & \begin{bmatrix} 1 & 4 & -3 & \vdots & a \\ 2 & 6 & 5 & \vdots & b \\ 1 & 6 & -14 & \vdots & c \end{bmatrix} \xrightarrow{-2R_1+R_2, -R_1+R_3} \begin{bmatrix} 1 & 4 & -3 & \vdots & a \\ 0 & -2 & 11 & \vdots & -2a+b \\ 0 & 2 & -11 & \vdots & -a+c \end{bmatrix} \\ & \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 4 & -3 & \vdots & a \\ 0 & 1 & \frac{-11}{2} & \vdots & \frac{2a-b}{2} \\ 0 & 2 & -11 & \vdots & -a+c \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 4 & -3 & \vdots & a \\ 0 & 1 & \frac{-11}{2} & \vdots & \frac{2a-b}{2} \\ 0 & 0 & 0 & \vdots & -3a+b+c \end{bmatrix}. \end{aligned}$$

So the system is consistent if and only if $-3a + b + c = 0$.

Problem 6. Perform elementary row or column operations to find the determinant:

$$\begin{aligned} \det \begin{bmatrix} 3r & 3t & 3s \\ a-2u & c-2w & b-2v \\ u & w & v \end{bmatrix} & \stackrel{2R_3+R_2}{=} \det \begin{bmatrix} 3r & 3t & 3s \\ a & c & b \\ u & w & v \end{bmatrix} \stackrel{\frac{1}{3}R_1}{=} 3 \det \begin{bmatrix} r & t & s \\ a & c & b \\ u & w & v \end{bmatrix} \\ & \stackrel{R_1 \rightleftharpoons R_2}{=} -3 \det \begin{bmatrix} a & c & b \\ r & t & s \\ u & w & v \end{bmatrix} \stackrel{C_2 \rightleftharpoons C_3}{=} 3 \det \begin{bmatrix} a & b & c \\ r & s & t \\ u & v & w \end{bmatrix} = 3 \cdot 3 = 9. \end{aligned}$$

Problem 7. We have

$$\begin{aligned} \det(2B^3 A^{-1} B^T) &= \det(2B^3) \det(A^{-1}) \det(B^T) \\ &= 2^5 \det(B^3) \frac{1}{\det(A)} \det(B) \\ &= 2^5 \det(B)^3 \frac{1}{\det(A)} \det(B) \\ &= 32 \cdot (-1)^3 \cdot \frac{1}{2} \cdot (-1) \\ &= 16. \end{aligned}$$

Problem 8.

• True. Every *homogeneous* system in which the number of equations is less than the number of unknowns has infinitely many solutions (Theorem 1.2.1 of the book).

• False. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \quad BA = \begin{bmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{bmatrix}.$$

Therefore

$$\operatorname{tr}(AB) = (ae + bg) + (cf + dh) = (ea + fc) + (gb + hd) = \operatorname{tr}(BA).$$

- False. The matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

has the property $A^2 = A$, but $A \neq 0$ and $A \neq I$.

- True. Since $AB = 0$ and A^{-1} exists, we have

$$A^{-1}(AB) = A^{-1}0 \implies (A^{-1}A)B = 0 \implies IB = 0 \implies B = 0.$$