# Math 231 Practice Test 2 Solutions

1. (i)

$$\begin{aligned}\mathbf{u} \cdot (2\mathbf{u} - 3\mathbf{v}) &= (1,-1,0,2) \cdot ((2,-2,0,4) - (0,0,3,15)) \\ &= (1,-1,0,2) \cdot (2,-2,-3,-11) \\ &= (1)(2) + (-1)(-2) + (0)(-3) + (2)(-11) \\ &= -18.\end{aligned}$$

(ii)

$$\|\mathbf{u} + \mathbf{v}\| = \|(1,-1,1,7)\| = \sqrt{1+1+1+49} = \sqrt{52}.$$

(iii) The vector $\mathbf{u} - \mathbf{v}$ has coordinates $(1,-1,-1,-3)$ and norm $\sqrt{12}$. Hence the unit vector in the direction of $\mathbf{u} - \mathbf{v}$ is the vector $\frac{1}{\sqrt{12}}(1,-1,-1,-3) = (\frac{1}{\sqrt{12}}, \frac{-1}{\sqrt{12}}, \frac{-1}{\sqrt{12}}, \frac{-3}{\sqrt{12}})$.

(iv) The distance between $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\| = \sqrt{12}$.

(v)

$$\begin{aligned}\mathbf{u} \text{ orthogonal to } \mathbf{u} + a\mathbf{v} &\iff \mathbf{u} \cdot (\mathbf{u} + a\mathbf{v}) = 0 \\ &\iff (1,-1,0,2) \cdot (1,-1,a,2+5a) = 0 \\ &\iff (1)(1) + (-1)(-1) + (0)(a) + (2)(2+5a) = 0 \\ &\iff 6 + 10a = 0 \\ &\iff a = -\tfrac{3}{5}.\end{aligned}$$

2. If $f$ and $g$ are both odd functions, so is $f+g$ because

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = -(f+g)(x).$$

Similarly, if $f$ is an odd function and $\lambda$ is a scalar, the function $\lambda f$ is also odd:

$$(\lambda f)(-x) = \lambda f(-x) = -\lambda f(x) = -(\lambda f)(x).$$

It follows that the set of odd functions is closed under addition and scalar multiplication. Hence it must be a subspace of $\mathcal{F}(-\infty, +\infty)$.

3. $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$ if and only if the two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent. This means that $\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 = 0$ happens only when $\lambda_1 = \lambda_2 = 0$. Now

$$\begin{aligned}\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 = 0 &\iff \lambda_1(a\mathbf{v}_1 + b\mathbf{v}_2) + \lambda_2(c\mathbf{v}_1 + d\mathbf{v}_2) = 0 \\ &\iff (a\lambda_1 + c\lambda_2)\mathbf{v}_1 + (b\lambda_1 + d\lambda_2)\mathbf{v}_2 = 0 \\ &\iff a\lambda_1 + c\lambda_2 = b\lambda_1 + d\lambda_2 = 0 \quad (\text{since } \{\mathbf{v}_1, \mathbf{v}_2\} \text{ is a basis})\end{aligned}$$

It follows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent if and only if the homogeneous system

$$\begin{cases} a\lambda_1 + c\lambda_2 &= 0 \\ b\lambda_1 + d\lambda_2 &= 0 \end{cases}$$

has only the trivial solution $\lambda_1 = \lambda_2 = 0$. This happens precisely when the matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

is invertible, i.e., when $ad - bc \neq 0$.

4. We form the $4 \times 4$ matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and compute its determinant:

$$\det \begin{bmatrix} 1 & 1 & -2 & 2 \\ 1 & 0 & 1 & -2 \\ 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \quad \text{(expansion along second column)}$$

$$= -\left(\det \begin{bmatrix} 3 & 0 \\ 1 & 5 \end{bmatrix} - 2\det \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix}\right) \quad \text{(expansion along first column)}$$

$$= -(15 - 14) = -1$$

Since this determinant is non-zero, $\mathcal{B}$ is a basis for $\mathbb{R}^4$.

To find the coordinates of a generic vector $\mathbf{v} = (x, y, z, w)$ relative to $\mathcal{B}$, we have to express $\mathbf{v}$ as a linear combination $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 + d\mathbf{v}_4$ and find the coefficients $a, b, c, d$ in terms of $x, y, z, w$:

$$(x, y, z, w) = a(1, 1, 2, 0) + b(1, 0, 0, 0) + c(-2, 1, 3, 1) + d(2, -2, 0, 5)$$

$$\iff \begin{cases} a + b - 2c + 2d & = x \\ a + c - 2d & = y \\ 2a + 3c & = z \\ c + 5d & = w \end{cases} \iff \begin{bmatrix} 1 & 1 & -2 & 2 \\ 1 & 0 & 1 & -2 \\ 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

We solve this system by Gauss-Jordan elimination:

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 2 & x \\ 1 & 0 & 1 & -2 & y \\ 2 & 0 & 3 & 0 & z \\ 0 & 0 & 1 & 5 & w \end{array}\right] \xrightarrow{-R_1+R_2, -2R_1+R_3} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 2 & x \\ 0 & -1 & 3 & -4 & -x+y \\ 0 & -2 & 7 & -4 & -2x+z \\ 0 & 0 & 1 & 5 & w \end{array}\right]$$

$$\xrightarrow{-R_2, 2R_2+R_3} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 2 & x \\ 0 & 1 & -3 & 4 & x-y \\ 0 & 0 & 1 & 4 & -2y+z \\ 0 & 0 & 1 & 5 & w \end{array}\right] \xrightarrow{-R_3+R_4} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 2 & x \\ 0 & 1 & -3 & 4 & x-y \\ 0 & 0 & 1 & 4 & -2y+z \\ 0 & 0 & 0 & 1 & 2y-z+w \end{array}\right]$$

$$\xrightarrow{-4R_4+R_3, -4R_4+R_2, -2R_4+R_1} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 0 & x-4y+2z-2w \\ 0 & 1 & -3 & 0 & x-9y+4z-4w \\ 0 & 0 & 1 & 0 & -10y+5z-4w \\ 0 & 0 & 0 & 1 & 2y-z+w \end{array}\right]$$

$$\xrightarrow{3R_3+R_2, 2R_3+R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & x-24y+12z-10w \\ 0 & 1 & 0 & 0 & x-39y+19z-16w \\ 0 & 0 & 1 & 0 & -10y+5z-4w \\ 0 & 0 & 0 & 1 & 2y-z+w \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 15y-7z+6w \\ 0 & 1 & 0 & 0 & x-39y+19z-16w \\ 0 & 0 & 1 & 0 & -10y+5z-4w \\ 0 & 0 & 0 & 1 & 2y-z+w \end{array}\right]$$

So the system has the solution

$$a = 15y - 7z + 6w, b = x - 39y + 19z - 16w, c = -10y + 5z - 4w, d = 2y - z + w.$$

Thus the coordinates of $\mathbf{v} = (x, y, z, w)$ relative to $\mathcal{B}$ are

$$(x, y, z, w)_{\mathcal{B}} = (15y - 7z + 6w, x - 39y + 19z - 16w, -10y + 5z - 4w, 2y - z + w).$$

5. Let $W$ be the set of all symmetric $3 \times 3$ matrices, i.e., all $3 \times 3$ matrices $A$ such that $A^T = A$. Suppose $A, B$ are in $W$ and $\lambda$ is any scalar. Then

$$(A + B)^T = A^T + B^T = A + B \qquad \text{and} \qquad (\lambda A)^T = \lambda A^T = \lambda A.$$

Thus both $A + B$ and $\lambda A$ are in $W$. This shows $W$ is closed under addition and scalar multiplication, so it is a subspace of $\mathcal{M}_{3,3}$.

Now let us find a basis for $W$. Every matrix in $W$, being symmetric, has the form

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

for some real numbers $a, b, c, d, e, f$. We can decompose this matrix as

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we set

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that every matrix in $W$ is a linear combination of $\{A_1, \ldots, A_6\}$. In other words, $W = \text{span}\{A_1, \ldots, A_6\}$. On the other hand, $A_1, \ldots, A_6$ are linearly independent because if

$$aA_1 + bA_2 + cA_3 + dA_4 + eA_5 + fA_6 = 0$$

then

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies $a = b = c = d = e = f = 0$. Thus $\{A_1, \ldots, A_6\}$ is a basis for $W$. In particular, $\dim(W) = 6$.

6. (i) $W = \{(a, b, c) : a + 2b - c = 0\}$. In other words, $W$ consists of all vectors of the form $(a, b, a + 2b)$, where $a, b$ are arbitrary real numbers. Since

$$(a, b, a + 2b) = a(1, 0, 1) + b(0, 1, 2),$$

it follows that every vector in $W$ is a linear combination of $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (0, 1, 2)$. It is easy to see that $\mathbf{v}_1, \mathbf{v}_2$ are independent, so they must form a basis for $W$. In particular, $\dim(W) = 2$ (a plane through the origin).

(ii) $W = \{(a, b, c) : a = b = -c\}$. In other words, $W$ consists of all vectors of the form $(a, a, -a)$, where $a$ is an arbitrary real number. Since

$$(a, a, -a) = a(1, 1, -1),$$

it follows that every vector in $W$ is a scalar multiple of $\mathbf{v}_1 = (1, 1, -1)$ so $\mathbf{v}_1$ alone must form a basis for $W$. In particular, $\dim(W) = 1$ (a line through the origin).

7. (i) Augment the vector $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ to $A$ and reduce to row-echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & -2 & -3 & a \\ 1 & 4 & 3 & -1 & -4 & b \\ 2 & 3 & -4 & -7 & -3 & c \\ 3 & 8 & 1 & -7 & -8 & d \end{array}\right] \xrightarrow{-R_1+R_2, -2R_1+R_3, -3R_1+R_4} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -2 & -3 & a \\ 0 & 1 & 2 & 1 & -1 & -a+b \\ 0 & -3 & -6 & -3 & 3 & -2a+c \\ 0 & -1 & -2 & -1 & 1 & -3a+d \end{array}\right]$$

$$\xrightarrow{3R_2+R_3, R_2+R_4} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -2 & -3 & a \\ 0 & 1 & 2 & 1 & -1 & -a+b \\ 0 & 0 & 0 & 0 & 0 & -5a+3b+c \\ 0 & 0 & 0 & 0 & 0 & -4a+b+d \end{array}\right]$$

If follows that

$$A\mathbf{x} = \mathbf{b} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ is consistent } \iff -5a + 3b + c = -4a + b + d = 0.$$

(ii) We already have the row-echelon form $R$ of $A$:

$$R = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for $\text{Row}(A)$ will consist of the non-zero rows of $R$:

$$\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 1 & -1 \end{bmatrix}.$$

Since the leading 1's in $R$ occur along the first and second columns, a basis for $\text{Col}(A)$ will consist of the first and second columns of $A$:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 4 \\ 3 \\ 8 \end{bmatrix}.$$

Finally, to find a basis for Null($A$), let us solve the system $A\mathbf{x} = 0$, or equivalently, $R\mathbf{x} = 0$:

$$\begin{cases} x_1 + 3x_2 + x_3 - 2x_4 - 3x_5 & = 0 \\ x_2 + 2x_3 + x_4 - x_5 & = 0 \end{cases}$$

The leading variables are $x_1, x_2$ and the free variables are $x_3, x_4, x_5$. So the solutions of the system are obtained by assigning arbitrary values to $x_3, x_4, x_5$ and solving for $x_1, x_2$:

$$\begin{cases} x_1 & = 5r + 5s \\ x_2 & = -2r - s + t \\ x_3 & = r \\ x_4 & = s \\ x_5 & = t \end{cases} \qquad (r, s, t \text{ are arbitrary})$$

Thus

$$\mathbf{x} \text{ is in Null}(A) \iff \mathbf{x} = \begin{bmatrix} 5r + 5s \\ -2r - s + t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that a basis for Null($A$) is given by the vectors

$$\begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(iii) $\text{rank}(A) = \dim \text{Row}(A) = \dim \text{Col}(A) = 2$ and $\text{nullity}(A) = \dim \text{Null}(A) = 3$, so $\text{rank}(A) + \text{nullity}(A) = 5$.