

## Math 231 Final Review Sheet, 12/10/2004

Here is an itemized list of what could potentially be on the final exam. Make sure you know enough about each item.

- Solving systems of linear equations by Gaussian elimination; reducing a matrix to a row-echelon form.
- Addition and multiplication of matrices; the inverse of a matrix; elementary matrices; finding the inverse of a matrix by elementary row operations;  $A\mathbf{x} = \mathbf{b}$  has the unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$  when  $A$  is an invertible matrix; checking consistency of  $A\mathbf{x} = \mathbf{b}$  in general by reducing  $A$  to row-echelon form.
- Trace and transpose of a matrix; diagonal, upper and lower triangular, and symmetric matrices.
- Definition of  $\det(A)$  for a square matrix  $A$ ; determinant of triangular matrices; the effect of elementary row operations on the determinant; basic properties of determinant; a square matrix is invertible if and only if its determinant is non-zero; evaluating determinants by cofactor expansions; adjoint of a matrix; finding the inverse using the adjoint; Cramer's rule.
- Vectors in  $\mathbb{R}^n$ ; geometric interpretation in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ ; vector addition and scalar multiplication in  $\mathbb{R}^n$ ; norm of a vector in  $\mathbb{R}^n$ ; dot product and its basic properties.
- General notion of a vector space; basic examples:  $\mathbb{R}^n$ ,  $\mathcal{M}_{m,n}$ ,  $\mathcal{P}_n$ ,  $\mathcal{F}[a, b]$ ; subspaces of vector spaces; how to verify that a subset of a vector space is a subspace.
- Linear combinations; span of a set of vectors; linear dependence and independence; geometric meaning of dependence and independence in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ ; definition of a basis; coordinates of a vector relative to a basis; definition of dimension;  $\dim(\mathbb{R}^n) = n$ ,  $\dim(\mathcal{M}_{m,n}) = mn$ ,  $\dim(\mathcal{P}_n) = n + 1$ ; any collection of more than  $n$  vectors in  $\mathbb{R}^n$  is linearly dependent; no collection of less than  $n$  vectors in  $\mathbb{R}^n$  can span  $\mathbb{R}^n$ ; a collection of  $n$  vectors in  $\mathbb{R}^n$  is linearly independent if and only if it spans  $\mathbb{R}^n$ ; the determinant criterion for verifying that  $n$  given vectors in  $\mathbb{R}^n$  form a basis.
- The row space, column space and null space of a matrix; how to find bases for these spaces by reducing the matrix to row-echelon form; rank and nullity of a matrix; the rank plus nullity theorem for matrices.
- Inner products on vector spaces; norm induced by an inner product; orthogonal vectors; Cauchy-Schwarz Inequality; Pythagorean Theorem in inner product spaces; Orthogonal/orthonormal bases; coordinates of a vector relative to an orthogonal/orthonormal basis; the Gram-Schmidt orthogonalization process.
- Eigenvalues and eigenvectors of a square matrix; finding eigenvalues as roots of the characteristic polynomial; eigenvalues of a triangular matrix; finding eigenvectors and more generally a basis for each eigenspace.
- Diagonalizable matrices; an  $n \times n$  matrix is diagonalizable if and only if it has  $n$  linearly independent eigenvectors; given a diagonalizable matrix  $A$  how to find an invertible matrix  $P$  such that  $P^{-1}AP$  is diagonal; an  $n \times n$  matrix with  $n$  distinct

eigenvalues is diagonalizable; application of diagonalization in finding powers of a matrix.

- Linear maps between vector spaces; any linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  is given by left multiplication by an  $m \times n$  matrix; linear maps are determined by their action on basis vectors; kernel and image (range) of a linear map; rank and nullity of a linear map; the rank plus nullity theorem for linear maps.

### Some sample problems

1. Consider the matrix  $A = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 5 & -1 & 6 & -2 \end{bmatrix}$ .

- For what vectors  $\mathbf{b} \in \mathbb{R}^3$  is the system  $A\mathbf{x} = \mathbf{b}$  consistent?
- Find bases for  $\text{Row}(A)$  and  $\text{Col}(A)$  and  $\text{Null}(A)$ .
- What is  $\text{rank}(A)$  and  $\text{nullity}(A)$ ?

2. Consider the matrix  $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$ .

- Find the eigenvalues of  $A$ .
- Find a basis for each eigenspace of  $A$ .
- Find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ .
- Find the matrix  $A^{10}$ .

3. Verify that the vectors

$$\mathbf{v}_1 = (1, 0, -5) \quad \mathbf{v}_2 = (-1, 1, 4) \quad \mathbf{v}_3 = (0, 1, 0).$$

form a basis for  $\mathbb{R}^3$ . If  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the linear map such that

$$T(\mathbf{v}_1) = (1, 2, 1) \quad T(\mathbf{v}_2) = (2, 0, -3) \quad T(\mathbf{v}_3) = (1, 1, 1),$$

find a formula for  $T(x, y, z)$ .

4. Use the Gram-Schmidt process to transform the basis

$$\mathbf{u}_1 = (1, 2, -1) \quad \mathbf{u}_2 = (1, 3, 0) \quad \mathbf{u}_3 = (4, 1, 0)$$

for  $\mathbb{R}^3$  into an orthonormal basis.

5. Let  $T : \mathbb{R}^3 \rightarrow \mathcal{M}_{2,2}$  be the map defined by

$$T(x, y, z) = \begin{bmatrix} 2x & 0 \\ 0 & y - z \end{bmatrix}.$$

- Show that  $T$  is a linear map.
- Find  $\ker(T)$  and  $\text{image}(T)$ .
- Find  $\text{rank}(T)$  and  $\text{nullity}(T)$  and verify that the “rank plus nullity” theorem holds.