Math 231 Final Review Sheet, 12/11/2005

Here is a list of the topics that may be on the final exam. Make sure you know enough about each item.

- Solving systems of linear equations by Gaussian elimination; reducing a matrix to row-echelon form.
- Addition and multiplication of matrices; the inverse of a matrix; elementary matrices; finding the inverse of a matrix by elementary row operations; the system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ when A is invertible; verifying consistency of $A\mathbf{x} = \mathbf{b}$ by reducing A to row-echelon form.
- Trace and transpose of a matrix; diagonal, triangular and symmetric matrices.
- Evaluating $\det(A)$ by cofactor expansion; determinant of triangular matrices; the effect of elementary row operations on the determinant; basic properties of determinant; a square matrix is invertible if and only if its determinant is non-zero; adjoint of a matrix; finding the inverse using the adjoint; Cramer's rule.
- Vectors in $\mathbb{R}^n$; geometric interpretation of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$; vector addition and scalar multiplication in $\mathbb{R}^n$; norm of a vector in $\mathbb{R}^n$; dot product and its basic properties.
- General notion of a vector space; basic examples: $\mathbb{R}^n$, $\mathcal{M}_{m,n}$, $\mathcal{P}_n$, $\mathcal{F}[a, b]$; subspaces of vector spaces; how to verify that a subset of a vector space is a subspace.
- Linear combinations; span of a set of vectors; linear dependence and independence; geometric meaning of dependence and independence in $\mathbb{R}^2$ and $\mathbb{R}^3$; definition of a basis; coordinates of a vector relative to a basis; definition of dimension; $\dim(\mathbb{R}^n) = n$, $\dim(\mathcal{M}_{m,n}) = mn$, $\dim(\mathcal{P}_n) = n + 1$; any collection of more than n vectors in $\mathbb{R}^n$ is linearly dependent; no collection of less than n vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$; a collection of n vectors in $\mathbb{R}^n$ is linearly independent if and only if it spans $\mathbb{R}^n$; determinant criterion for when n given vectors in $\mathbb{R}^n$ form a basis.
- Row space, column space and null space of a matrix; finding bases for these spaces by reducing the matrix to row-echelon form; rank and nullity of a matrix; the rank plus nullity theorem for matrices.
- Inner products on vector spaces; norm induced by an inner product; orthogonal vectors; Cauchy-Schwarz Inequality; Pythagorean Theorem in inner product spaces; orthogonal/orthonormal bases; coordinates of a vector relative to an orthogonal/orthonormal basis; the Gram-Schmidt orthogonalization process.
- Eigenvalues and eigenvectors of a square matrix; finding eigenvalues as roots of the characteristic equation; eigenvalues of a triangular matrix; finding eigenvectors and more generally a basis for each eigenspace.
- Diagonalization of matrices; an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors; given a diagonalizable matrix A how to find an invertible matrix P such that $P^{-1}AP$ is diagonal; an $n \times n$ matrix with n distinct eigenvalues is diagonalizable; application of diagonalization in finding powers of a matrix.
- Linear maps between vector spaces; any linear map $\mathbb{R}^n \to \mathbb{R}^m$ is given by left multiplication by an $m \times n$ matrix; linear maps are determined by their action on basis vectors; kernel and image (range) of a linear map; rank and nullity of a linear map; the rank plus nullity theorem for linear maps.

Some sample problems

1. Consider the matrix $A = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 5 & -1 & 6 & -2 \end{bmatrix}$.

   (i) For what vectors $\mathbf{b}$ in $\mathbb{R}^3$ is the system $A\,\mathbf{x} = \mathbf{b}$ consistent?
   (ii) Find bases for $\mathrm{Row}(A)$ and $\mathrm{Col}(A)$ and $\mathrm{Null}(A)$.
   (iii) What is $\mathrm{rank}(A)$ and $\mathrm{nullity}(A)$?

2. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$.

(i) Find the eigenvalues of A.
(ii) Find a basis for each eigenspace of A.
(iii) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
(iv) Find the matrix A^{10}.

3. Verify that the vectors

$$\mathbf{v}_1 = (1, 0, -5) \qquad \mathbf{v}_2 = (-1, 1, 4) \qquad \mathbf{v}_3 = (0, 1, 0).$$

form a basis for $\mathbb{R}^3$. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear map such that

$$T(\mathbf{v}_1) = (1, 2, 1) \qquad T(\mathbf{v}_2) = (2, 0, -3) \qquad T(\mathbf{v}_3) = (1, 1, 1),$$

find a formula for $T(x, y, z)$.

4. Use the Gram-Schmidt process to transform the basis

$$\mathbf{u}_1 = (1, 2, -1) \qquad \mathbf{u}_2 = (1, 3, 0) \qquad \mathbf{u}_3 = (4, 1, 0)$$

for $\mathbb{R}^3$ into an orthonormal basis.

5. Suppose λ_1 and λ_2 are the eigenvalues of a 2×2 matrix A. Show that $\lambda_1 + \lambda_2 = \mathrm{tr}(A)$ and $\lambda_1\lambda_2 = \det(A)$.

6. Let $T : \mathbb{R}^3 \to \mathcal{M}_{2,2}$ be the map defined by

$$T(x, y, z) = \begin{bmatrix} 2x & 0 \\ 0 & y - z \end{bmatrix}.$$

(i) Show that T is a linear map.
(ii) Find $\ker(T)$ and $\mathrm{image}(T)$.
(iii) Find $\mathrm{nullity}(T)$ and $\mathrm{rank}(T)$ and verify that the "rank plus nullity" theorem holds.