Math 320 Homework 10

Due Tuesday December 2, 2003

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ -x+2 & \text{if } x > 1 \end{cases}$$

Prove that f is not differentiable at $x = 1$.

Problem 2. Suppose f and g are differentiable on $\mathbb{R}$, $f(a) = g(a)$ and $f'(x) \leq g'(x)$ for all $x \geq a$. Show that $f(x) \leq g(x)$ for all $x \geq a$. Give a physical interpretation of this result. (Hint: Consider the function $f(x) - g(x)$.)

Problem 3. What is wrong with the following "proof" of the Cauchy Mean Value Theorem (Lay's Theorem 27.1)?

"By the ordinary Mean Value Theorem, we have $f(b) - f(a) = (b-a)f'(c)$ and $g(b) - g(a) = (b-a)g'(c)$ for a c bewteen a and b. Multiplying the two equations, we obtain $(f(b) - f(a))(b-a)g'(c) = (g(b) - g(a))(b-a)f'(c)$. Since $a \neq b$, we can cancel $b - a$ from both sides to obtain $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$."

Problem 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function which satisfies

$$|f(x) - f(y)| \leq C\,|x-y|^2 \qquad \text{for all } x, y \in \mathbb{R}.$$

Here C is some positive constant. Show that f is a constant function. (Hint: Verify that $f'(x)$ exists and is zero for all x.)

Problem 5. Find the following limits:

$$\lim_{x \to 0} \frac{\sin(x^2)}{1 - \cos x} \qquad\qquad \lim_{x \to e} \frac{\log x - 1}{x}$$

Problem 6. Let $f(x) = \sqrt{x}$.

(i) Find the first three derivatives of f.

(ii) Find the degree 2 Taylor polynomial P_2 of f about the point 4. Write your answer in powers of $x - 4$.

(iii) Use Taylor's Theorem to show that

$$|\sqrt{x} - P_2(x)| \leq \frac{1}{16}\,|x-4|^3 \quad \text{for all } x \geq 1.$$

(iv) Use P_2 to find the approximate value of $\sqrt{4.1}$. What is the error in this approximation (use a calculator)? Is this consistent with (iii)?