Math 320 Homework 10 solutions

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ -x+2 & \text{if } x > 1 \end{cases}$$

Prove that f is not differentiable at $x = 1$.

We have

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{-x + 2 - 1}{x - 1} = -1$$

and

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} x + 1 = 2$$

Thus the limit of $(f(x) - f(1))/(x - 1)$ as $x \to 1$ does not exist.

Problem 2. Suppose f and g are differentiable on $\mathbb{R}$, $f(a) = g(a)$ and $f'(x) \leq g'(x)$ for all $x \geq a$. Show that $f(x) \leq g(x)$ for all $x \geq a$. Give a physical interpretation of this result.

The function $h = f - g$ satisfies $h'(x) = f'(x) - g'(x) \leq 0$ for all $x \geq a$. Hence h is decreasing on $[a, +\infty)$. Since $h(a) = 0$, we must have $h(x) \leq 0$ for all $x \geq a$. It follows that $f(x) \leq g(x)$ for all $x \geq a$.

If you think of x as "time" and $f(x)$ and $g(x)$ as the "positions" at time x of two objects moving along a straight line, the above statement says that if f and g are neck and neck at $x = a$ and if g moves faster than f, then g will be ahead of f at all times after a.

Problem 3. What is wrong with the following "proof" of the Cauchy Mean Value Theorem (Lay's Theorem 27.1)?

"By the ordinary Mean Value Theorem, we have $f(b) - f(a) = (b - a)f'(c)$ and $g(b) - g(a) = (b - a)g'(c)$ for a c bewteen a and b. Multiplying the two equations, we obtain $(f(b) - f(a))(b - a)g'(c) = (g(b) - g(a))(b - a)f'(c)$. Since $a \neq b$, we can cancel $b - a$ from both sides to obtain $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$."

The problem is that the point c may be different for f and g; that is, we have

$$f(b) - f(a) = (b - a)f'(c_1) \qquad \text{for some } c_1 \in (a, b)$$

and

$$g(b) - g(a) = (b - a)g'(c_2) \qquad \text{for some } c_2 \in (a, b)$$

and it may happen that $c_1 \neq c_2$.

Problem 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function which satisfies

$$|f(x) - f(y)| \leq C\,|x - y|^2 \qquad \text{for all } x, y \in \mathbb{R}.$$

Here C is some positive constant. Show that f is a constant function.

Fix a point p and let $x \neq p$. Then

$$\left|\frac{f(x)-f(p)}{x-p}\right| = \frac{|f(x)-f(p)|}{|x-p|} \leq \frac{C|x-p|^2}{|x-p|} = C|x-p|.$$

As $x \to p$, we obtain

$$\lim_{x\to p} \frac{f(x)-f(p)}{x-p} = 0.$$

This shows that $f'(p)$ exists and is 0. Since p was arbitrary, it follows that f is a constant function.

Problem 5. Find the following limits:

- $\lim_{x\to 0} \dfrac{\sin(x^2)}{1-\cos x}$.

Note that $\lim_{x\to 0} x/\sin x = 1$. Hence

$$\lim_{x\to 0} \frac{(\sin(x^2))'}{(1-\cos x)'} = \lim_{x\to 0} \frac{2x\cos(x^2)}{\sin x} = 2$$

It follows from l'Hospital that

$$\lim_{x\to 0} \frac{\sin(x^2)}{1-\cos x} = 2.$$

- $\lim_{x\to e} \dfrac{\log x - 1}{x}$.

Since $\lim_{x\to e} \log x - 1 = 0$ and $\lim_{x\to e} x = e$, we have

$$\lim_{x\to e} \frac{\log x - 1}{x} = 0.$$

Problem 6. Let $f(x) = \sqrt{x}$.

(i) Find the first three derivatives of f.

$f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, and $f'''(x) = \frac{3}{8}x^{-5/2}$.

(ii) Find the degree 2 Taylor polynomial P_2 of f about the point 4. Write your answer in powers of $x-4$.

$$P_2(x) = f(4) + f'(4)(x-4) + \frac{1}{2}f''(4)(x-4)^2$$

which simplifies to

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2.$$

(iii) Use Taylor's Theorem to show that

$$|\sqrt{x} - P_2(x)| \leq \frac{1}{16}\,|x-4|^3 \quad \text{for all } x \geq 1.$$

By Taylor's Theorem,

$$\sqrt{x} - P_2(x) = \frac{1}{6} f'''(c)(x-4)^3$$

for some c between x and 4. If $x \geq 1$, we have $c > 1$, so that

$$|f'''(c)| = \frac{3}{8}|c^{-5/2}| \leq \frac{3}{8}.$$

It follows that

$$|\sqrt{x} - P_2(x)| \leq \frac{1}{6} \cdot \frac{3}{8}\,|x-4|^3 = \frac{1}{16}\,|x-4|^3.$$

(iv) Use P_2 to find the approximate value of $\sqrt{4.1}$. What is the error in this approximation (use a calculator)? Is this consistent with (iii)?

We have

$$\sqrt{4.1} \approx P_2(4.1) = 2 + \frac{1}{4}(0.1) - \frac{1}{64}(0.1)^2 = 2.0248438.$$

The actual value of $\sqrt{4.1}$ is 2.0248457. This is consistent with (iii) since $|2.0248457 - 2.0248438| = 0.0000019$ is smaller than $\frac{1}{16} \cdot (0.1)^3 = 0.0000625$.