Math 320 Homework 11

Due Thursday December 11, 2003

Problem 1. Give an example of a function $f : [0,1] \to \mathbb{R}$ which is *not* integrable, such that $|f| : [0,1] \to \mathbb{R}$ is integrable.

Problem 2. Let $f(x) = x$ on $[a,b]$. We know from calculus that $\int_a^b f(x)\,dx = (b^2 - a^2)/2$. This exercise shows you how to obtain the same result from the definition of integral.

(i) Let $n \in \mathbb{N}$ and P be the partition $\{x_0 = a, x_1, \ldots, x_n = b\}$ for which $x_i - x_{i-1} = (b-a)/n$ for every $1 \le i \le n$. Find

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad \text{and} \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

(ii) Using (i), show that

$$L(f,P) = a(b-a) + \frac{n(n-1)}{2} \left(\frac{b-a}{n} \right)^2$$

and

$$U(f,P) = a(b-a) + \frac{n(n+1)}{2} \left(\frac{b-a}{n} \right)^2.$$

(iii) By taking limit as $n \to \infty$ in (ii), show that

$$\overline{\int_a^b} f(x)\,dx \le \frac{b^2 - a^2}{2} \le \underline{\int_a^b} f(x)\,dx.$$

Conclude that $\int_a^b f(x)\,dx$ exists and is equal to $(b^2 - a^2)/2$.

Problem 3. Let $f : [a,b] \to \mathbb{R}$ be a continuous function which satisfies $f(x) \ge 0$ for all $x \in [a,b]$. If $\int_a^b f(x)\,dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

Problem 4. (Mean Value Theorem for Integrals) Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists a point $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

The quantity on the right is often called the *average value* of f over $[a,b]$. Interpret this result geometrically. (Hint: Let $f(s) = \inf_{x \in [a,b]} f(x)$ and $f(t) = \sup_{x \in [a,b]} f(x)$ and note that

$$f(s) \le \frac{1}{b-a} \int_a^b f(x)\,dx \le f(t).)$$