Math 320 Homework 11 solutions

Problem 1. Give an example of a function $f : [0,1] \to \mathbb{R}$ which is *not* integrable, such that $|f| : [0,1] \to \mathbb{R}$ is integrable.

Take $f(x) = 1$ if $x \in [0,1] \cap \mathbb{Q}$ and $f(x) = -1$ if $x \in [0,1] \setminus \mathbb{Q}$. Then f is not integrable on $[0,1]$ since for any partition P of $[0,1]$, $L(f,P) = -1$ and $U(f,P) = 1$. On the other hand, $|f|$ is the constant function 1, which is certainly integrable.

Problem 2. Let $f(x) = x$ on $[a,b]$. We know from calculus that $\int_a^b f(x)\,dx = (b^2 - a^2)/2$. This exercise shows you how to obtain the same result from the definition of integral.

(i) Let $n \in \mathbb{N}$ and P be the partition $\{x_0 = a, x_1, \ldots, x_n = b\}$ for which $x_i - x_{i-1} = (b-a)/n$ for every $1 \le i \le n$. Find

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad \text{and} \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

For simplicity, set $\Delta x = (b-a)/n$. Then $x_i = a + i\Delta x$. Since f is increasing, we have

$$m_i = f(x_{i-1}) = x_{i-1} = a + (i-1)\Delta x$$

and

$$M_i = f(x_i) = x_i = a + i\Delta x.$$

(ii) Using (i), show that

$$L(f,P) = a(b-a) + \frac{n(n-1)}{2}\left(\frac{b-a}{n}\right)^2$$

and

$$U(f,P) = a(b-a) + \frac{n(n+1)}{2}\left(\frac{b-a}{n}\right)^2.$$

We have

$$\begin{aligned} L(f,P) = \sum_{i=1}^n m_i \Delta x = \Delta x \sum_{i=1}^n (a + (i-1)\Delta x) \\ &= na\Delta x + (\Delta x)^2 \sum_{i=1}^n (i-1) \\ &= na\Delta x + (\Delta x)^2 \frac{n(n-1)}{2} \\ &= a(b-a) + \frac{n(n-1)}{2}\left(\frac{b-a}{n}\right)^2. \end{aligned}$$

Similarly,

$$
\begin{aligned}
U(f,P)=\sum_{i=1}^{n} M_i\Delta x &= \Delta x\sum_{i=1}^{n}(a+i\Delta x)\\
&= na\Delta x+(\Delta x)^2\sum_{i=1}^{n} i\\
&= na\Delta x+(\Delta x)^2\frac{n(n+1)}{2}\\
&= a(b-a)+\frac{n(n+1)}{2}\left(\frac{b-a}{n}\right)^2.
\end{aligned}
$$

(iii) By taking limit as $n\to\infty$ in (ii), show that

$$\overline{\int_a^b} f(x)\,dx \le \frac{b^2-a^2}{2} \le \underline{\int_a^b} f(x)\,dx.$$

Conclude that $\int_a^b f(x)\,dx$ exists and is equal to $(b^2-a^2)/2$.

It follows from (ii) that

$$\overline{\int_a^b} f(x)\,dx \le U(f,P)=a(b-a)+\frac{n(n+1)}{2}\left(\frac{b-a}{n}\right)^2$$

Since this holds for every $n\in\mathbb{N}$, we must have

$$\overline{\int_a^b} f(x)\,dx \le \lim_{n\to\infty}\left[a(b-a)+\frac{n(n+1)}{2}\left(\frac{b-a}{n}\right)^2\right]=a(b-a)+\frac{(b-a)^2}{2}=\frac{b^2-a^2}{2}.$$

Similarly, since

$$\underline{\int_a^b} f(x)\,dx \ge L(f,P)=a(b-a)+\frac{n(n-1)}{2}\left(\frac{b-a}{n}\right)^2$$

for every $n\in\mathbb{N}$, we must have

$$\underline{\int_a^b} f(x)\,dx \ge \lim_{n\to\infty}\left[a(b-a)+\frac{n(n-1)}{2}\left(\frac{b-a}{n}\right)^2\right]=a(b-a)+\frac{(b-a)^2}{2}=\frac{b^2-a^2}{2}.$$

This proves the required double inequality. Since $\underline{\int_a^b}\le\overline{\int_a^b}$, we must have

$$\overline{\int_a^b} f(x)\,dx=\underline{\int_a^b} f(x)\,dx=\int_a^b f(x)\,dx=\frac{b^2-a^2}{2}.$$

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function which satisfies $f(x) \geq 0$ for all $x \in [a, b]$. If $\int_a^b f(x)\,dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

It suffices by continuity to prove that $f(x) = 0$ for all $x \in (a, b)$. Assume by way of contradiction that $f(p) > 0$ for some $p \in (a, b)$. Since f is continuous at p, there is an interval $[p-r, p+r] \subset [a, b]$ such that $f(x) \geq \frac{1}{2} f(p)$ for all $x \in [p-r, p+r]$. Then

$$\int_a^b f(x)\,dx = \left(\int_a^{p-r} + \int_{p-r}^{p+r} + \int_{p+r}^b\right) f(x)\,dx \geq \int_{p-r}^{p+r} f(x)\,dx$$

simply because $f \geq 0$. This gives

$$\int_a^b f(x)\,dx \geq \int_{p-r}^{p+r} f(x)\,dx \geq \frac{1}{2} f(p) \cdot 2r = r f(p) > 0$$

which contradicts $\int_a^b f(x)\,dx = 0$.

Problem 4. (Mean Value Theorem for Integrals) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists a point $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

The quantity on the right is often called the *average value* of f over $[a, b]$. Interpret this result geometrically.

Since f is continuous on the compact interval $[a, b]$, it assumes its maximum and minimum. Let $f(s) = m = \inf_{x\in[a,b]} f(x)$ and $f(t) = M = \sup_{x\in[a,b]} f(x)$. If $m = M$, then f is constant and the result is trivial. Otherwise $m < M$ and so f is non-constant. Since $M - f(x) \geq 0$ and $M - f$ is not identically zero, the previous problem implies $\int_a^b (M - f(x))\,dx > 0$ or $\int_a^b f(x)\,dx < M(b-a)$. Similarly, since $f(x) - m \geq 0$ and $f - m$ is not identically zero, we have $\int_a^b (f(x) - m)\,dx > 0$ or $m(b-a) < \int_a^b f(x)\,dx$. It follows that

$$f(s) < \frac{1}{b-a} \int_a^b f(x)\,dx < f(t).$$

Therefore, by the Intermediate Value Theorem, there exists a $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Geometrically, the theorem says that for some $c \in (a, b)$, the area of the rectangle of width $b - a$ and hight $f(c)$ is equal to the area under the graph of f from a to b.