Math 319/320 Homework 2

Due Thursday September 18, 2003

Problem 1. Prove the identity: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Problem 2. Let $f : A \to B$ be a function. Suppose that C and $\{C_j, j \in \mathbb{N}\}$ are subsets of A and D is a subset of B. Are the following statements true or false? Justify your answers by a brief proof or a counterexample.

(i) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

(ii) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

(iii) If f is injective, then $f(\bigcap_{j\in\mathbb{N}} C_j) = \bigcap_{j\in\mathbb{N}} f(C_j)$.

(iv) If f is surjective, then $f(\bigcap_{j\in\mathbb{N}} C_j) = \bigcap_{j\in\mathbb{N}} f(C_j)$.

Problem 3. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that the composition $g \circ f$ is injective. Is f necessarily injective? What about g? Give brief proofs or counterexamples.

Problem 4. We showed in class that the set of *positive* rational numbers is denumerable. Using this, deduce that the set of *all* rational numbers is denumerable. Give a complete argument that is based on Definition 8.6; do not quote other results from the book without proof.

Bonus Problem. Enumerate the set $\mathbb{Q} \cap [0, 1]$ of rational numbers between 0 and 1 as $q_1, q_2, \ldots, q_j, \ldots$. Find numbers $r_j > 0$ such that

$$[0, 1] \not\subset \bigcup_{j=1}^{\infty} (q_j - r_j, q_j + r_j).$$

This illustrates a surprising fact: The rationals are everywhere "dense" in the real numbers. Nevertheless, when we thicken them up by including each into a little interval, we still may not capture all the reals.