Math 319/320 Homework 2: solutions

Problem 1. Prove the identity: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

(i) To show $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$:
Suppose that $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$, i.e. $x \notin B$ and $x \notin C$. Therefore $x \in A \setminus B$ and $x \in A \setminus C$. Therefore $x \in (A \setminus B) \cap (A \setminus C)$.

(ii) To show $(A \setminus B) \cap (A \setminus C) \subset A \setminus (B \cup C)$:
Suppose that $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A$ but $x \notin B$ and $x \notin C$. Therefore $x \notin B \cup C$. Therefore $x \in A \setminus (B \cup C)$.

Problem 2. Let $f : A \to B$ be a function and suppose that $C, C_j, j \geq 1$, are subsets of A and D is a subset of B. Are the following statements true or false? Justify your answers by a brief proof or a counterexample.

(i) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

This is FALSE: Suppose that $A = \mathbb{R}$, $C = (-\infty, 0]$, and $f(x) = x^2$. Then $f(A) = f(C) = [0, \infty)$. So

$$f(A \setminus C) = (0, \infty) \neq f(A) \setminus f(C) = \emptyset.$$

(ii) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

This is TRUE. Proof:
Suppose $x \in f^{-1}(B \setminus D)$. Then $f(x) \in B \setminus D$, so that $f(x) \in B$ and $f(x) \notin D$. Therefore $x \in f^{-1}(B)$ and $x \notin f^{-1}(D)$. Therefore $x \in f^{-1}(B) \setminus f^{-1}(D)$.
Conversely, suppose that $x \in f^{-1}(B) \setminus f^{-1}(D)$. Then $f(x) \in B$ and $f(x) \notin D$. Therefore $f(x) \in B \setminus D$. So $x \in f^{-1}(B \setminus D)$.

(iii) If f is injective then $f(\cap_{j \geq 1} C_j) = \cap_{j \geq 1} f(C_j)$.

This is TRUE. Proof:
$f(\cap_{j \geq 1} C_j) \subset \cap_{j \geq 1} f(C_j)$ for any f. (proof left to you – it's easy.)
The converse holds only for injective f. To see this, suppose that $y \in \cap_{j \geq 1} f(C_j)$. Then for all j, $y \in f(C_j)$. Therefore for all j there is $c_j \in C_j$ such that $f(c_j) = y$. Since f is injective and $f(c_j) = f(c_1)$ for all j, we must have $c_j = c_1$ for all j. Therefore $c_1 = c_j \in C_j$ for all j. So $c_1 \in \cap_{j \geq 1} C_j$ and $y = f(c_1) \in f(\cap_{j \geq 1} C_j)$.

(iv) If f is surjective then $f(\cap_{j \geq 1} C_j) = \cap_{j \geq 1} f(C_j)$.
This is FALSE. Take $A = B = \mathbb{R}$, $f(x) = x^2$, $C_1 = \{-1\}$ and $C_j = \{1\}$ for all $j \geq 2$. Then $\cap_{j \geq 1} C_j = \emptyset$, while $\cap_{j \geq 1} f(C_j) = \{1\}$.

Problem 3. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that the composite $g \circ f$ are injective. Is f necessarily injective? What about g? Give brief proofs or counterexamples.

f must be injective: Proof by contradiction.

We assume that f is not injective and find a contradiction.

If f is not injective then there are distinct points $x \neq y \in A$ such that $f(x) = f(y)$. But then

$$g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y).$$

Since $g \circ f$ is injective, this means that $x = y$, a contradiction.

g need not be injective: Example:

Let $A = \{1\}$, $B = C = \{1, 2\}$ and define f by: $f(1) = 1$, and let $g(1) = g(2) = 1$. Then $g \circ f$ has to be injective since A has just one point, but g is not!