Math 319/320 Homework 4

Due Thursday October 2, 2003

Recap: For a given set $S \subset \mathbb{R}$, we define

- the *interior* of S as
$$\text{int}(S) = \{x : \text{ some neighborhood of } x \text{ is contained in } S\}$$
- the *accumulation set* of S as
$$S' = \{x : \text{every neighborhood of } x \text{ contains a point of } S \text{ other than } x\}$$
- the *closure* of S as
$$\text{cl}(S) = S \cup S'$$
- the *boundary* of S as
$$\{x : \text{every neighborhood of } x \text{ intersects both } S \text{ and } \mathbb{R} \setminus S\}$$

We have the following facts:

- S is open $\iff \mathbb{R} \setminus S$ is closed $\iff S = \text{int}(S)$
- S is closed $\iff \mathbb{R} \setminus S$ is open $\iff S = \text{cl}(S)$.
- Arbitrary unions and finite intersections of open sets are open. Arbitrary intersections and finite unions of closed sets are closed.

Problem 1. Classify the following subsets of $\mathbb{R}$ as open, closed, neither open nor closed, or both open and closed:

- $[-1, 1] \cup \{2\}$
- $\{x \in \mathbb{R} : \sin x \geq 0\}$
- $\bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right)$
- $\{x \in (0, 1) : x \text{ is irrational}\}$

Problem 2. Find $\text{int}(S)$, S', $\text{cl}(S)$ and $\text{bd}(S)$ for each set S in problem 1.

Problem 3. In each case, give an example of a non-empty set $S \subset \mathbb{R}$ with the corresponding property:

- $S = \text{bd}(S)$
- $S' = \text{bd}(S)$
- $\text{cl}(S) = \text{int}(S)$

Problem 4. Suppose S is a non-empty subset of $\mathbb{R}$ and x is an accumulation point of S. Show that every neighborhood of x contains infinitely many points of S. (Hint: What happens if some neighborhood of x contains only finitely many points of S?)

Bonus Problem. Does there exist a set $S \subset \mathbb{R}$, other than $\emptyset$ and $\mathbb{R}$, such that $\text{bd}(S) = \emptyset$?