

Math 319/320 Homework 4 solutions

Problem 1. Classify the following subsets of \mathbb{R} as open, closed, neither open nor closed, or both open and closed:

(i) $[-1, 1] \cup \{2\}$

CLOSED, since $[-1, 1]$ is closed and $\{2\}$ is closed, and the union of two closed sets is closed.

(ii) $\{x \in \mathbb{R} : \sin x \geq 0\}$

CLOSED. The complement of this set is $\bigcup_{n \in \mathbb{Z}} ((2n-1)\pi, 2n\pi)$. Since this complement is a union of open intervals, it is open, so the set itself is closed.

(iii) $\bigcup_{n=1}^{\infty} [\frac{1}{n+1}, \frac{1}{n})$

OPEN, since it is easy to verify that this union is just the open interval $(0, 1)$.

(iv) $\{x \in (0, 1) : x \text{ is irrational}\}$

NEITHER. This set is not open since for every irrational $x \in (0, 1)$ and every neighborhood $N(x, r)$, there are rational numbers in $N(x, r)$. The set is not closed either, since its closure is $[0, 1]$, which is a larger set (recall that a set is closed iff it coincides with its closure).

Problem 2. Find $\text{int}(S)$, S' , $\text{cl}(S)$ and $\text{bd}(S)$ for each set S in problem 1.

(i) If $S = [-1, 1] \cup \{2\}$, then $\text{int}(S) = (-1, 1)$, $S' = [-1, 1]$, $\text{cl}(S) = [-1, 1] \cup \{2\}$ and $\text{bd}(S) = \{-1, 1, 2\}$.

(ii) If $S = \{x \in \mathbb{R} : \sin x \geq 0\}$, then

$$\text{int}(S) = \{x \in \mathbb{R} : \sin x > 0\} = \bigcup_{n \in \mathbb{Z}} (2n\pi, (2n+1)\pi),$$

$S' = \text{cl}(S) = S$ and

$$\text{bd}(S) = \{x \in \mathbb{R} : \sin x = 0\} = \{n\pi : n \in \mathbb{Z}\}.$$

(iii) If $S = \bigcup_{n=1}^{\infty} [\frac{1}{n+1}, \frac{1}{n}) = (0, 1)$, then $\text{int}(S) = S$, $S' = \text{cl}(S) = [0, 1]$ and $\text{bd}(S) = \{0, 1\}$.

(iv) If $S = (0, 1) \setminus \mathbb{Q}$, then $\text{int}(S) = \emptyset$, $S' = \text{cl}(S) = \text{bd}(S) = [0, 1]$.

Problem 3. In each case, give an example of a non-empty set $S \subset \mathbb{R}$ with the corresponding property:

- $S = \text{bd}(S)$

Any finite set S will do.

- $S' = \text{bd}(S)$

Let $S = [0, 1] \cap \mathbb{Q}$. Then $S' = \text{bd}(S) = [0, 1]$.

- $\text{cl}(S) = \text{int}(S)$

Let $S = \mathbb{R}$. Then $\text{cl}(S) = \text{int}(S) = \mathbb{R}$.

Problem 4. Suppose S is a non-empty subset of \mathbb{R} and x is an accumulation point of S . Show that every neighborhood of x contains infinitely many points of S .

Assume by way of contradiction that some neighborhood $N(x, r)$ of x contains only finitely many points of S . Then the punctured neighborhood $N^*(x, r)$ also contains finitely many points of S . Let $N^*(x, r) \cap S = \{p_1, p_2, \dots, p_k\}$. Define $\delta = \min\{|x - p_j| : j = 1, \dots, k\} > 0$. Then it is easy to check that $N^*(x, \delta) \cap S = \emptyset$. This contradicts x being an accumulation point of S .

Bonus Problem. Does there exist a set $S \subset \mathbb{R}$, other than \emptyset and \mathbb{R} , such that $\text{bd}(S) = \emptyset$?

The answer is NO. For any set S such that $S \neq \emptyset$ and $S \neq \mathbb{R}$, we show that $\text{bd}(S) \neq \emptyset$. Choose some $a \in S$ and some $b \in \mathbb{R} \setminus S$. Without losing generality, assume that $a < b$ (the case $a > b$ is treated similarly). The set $S \cap [a, b]$ is non-empty (it contains a) and bounded above (by b) so it has a least upper bound $c = \sup(S \cap [a, b])$. Note that $a \leq c \leq b$. We claim that $c \in \text{bd}(S)$.

Suppose there is a neighborhood $N(c, r)$ which is contained in S . Then $c + r \leq b$ since otherwise $c \leq b < c + r$ and we would have $b \in S$. It follows that $c + \frac{r}{2}$ belongs to $S \cap [a, b]$, which contradicts $c = \sup(S \cap [a, b])$.

Now suppose there is a neighborhood $N(c, r)$ which is contained in $\mathbb{R} \setminus S$. Then $a \leq c - r$ since otherwise $c - r < a \leq c$ and we would have $a \notin S$. It follows that $c - \frac{r}{2}$ is an upper bound for $S \cap [a, b]$, which is a contradiction since c is the least upper bound.

Summing up, we have shown that every neighborhood of c must intersect both S and $\mathbb{R} \setminus S$. This implies $c \in \text{bd}(S)$.