

Math 320 Homework 5
Due Thursday October 16, 2003

Recall the following definitions:

- A set $S \subset \mathbb{R}$ has the *Bolzano-Weierstrass (BW) property* if every infinite subset of S has an accumulation point in S .
- A collection $\{U_i\}$ of open sets in \mathbb{R} is an *open cover* of S if $S \subset \bigcup_i U_i$. We also say that $\{U_i\}$ *covers* S . Any sub-collection of $\{U_i\}$ which covers S is called a *subcover*.
- S is *compact* if every open cover of S has a finite subcover.

We have the following two theorems:

- **Bolzano-Weierstrass Theorem.** A set $S \subset \mathbb{R}$ has the BW property if and only if S is bounded and closed.
- **Heine-Borel Theorem.** A set $S \subset \mathbb{R}$ is compact if and only if it is bounded and closed.

Thus for subsets S of \mathbb{R} :

$$S \text{ is bounded and closed} \iff S \text{ has BW property} \iff S \text{ is compact}$$

Problem 1. Suppose $E = \{x_n : n \in \mathbb{N}\}$ is a sequence of points in \mathbb{R} such that $|x_n| \geq n$ for every $n \in \mathbb{N}$. Show that E does not have any accumulation point in \mathbb{R} . (Hint: Given $x \in \mathbb{R}$ and a neighborhood $N(x, r)$, show that $x_n \notin N(x, r)$ if $n > |x| + r$. Conclude that x is not an accumulation point of E .)

Problem 2. True or false? Justify with a brief proof or a counterexample.

- Every finite set is compact.
- The set $\{x \in \mathbb{R} : x - x^2 > 0\}$ is compact.
- The set $\mathbb{Q} \cap [0, 1]$ is compact.
- No open cover of $(-1, 1)$ can have a finite subcover.

Problem 3. Is it true that the union of any number of compact sets is compact? What about the union of a finite number of compact sets?

Problem 4. Suppose $S \subset \mathbb{R}$ is compact. Show that $\sup(S)$ and $\inf(S)$ exist and both belong to S .