Math 320 Homework 5 solutions

Problem 1. Suppose $E = \{x_n : n \in \mathbb{N}\}$ is a sequence of points in $\mathbb{R}$ such that $|x_n| \geq n$ for every $n \in \mathbb{N}$. Show that E does not have any accumulation point in $\mathbb{R}$.

Given $x \in \mathbb{R}$, a neighborhood $N(x, r)$ and an integer $n > |x| + r$, we have

$$|x_n - x| \geq |x_n| - |x| \geq n - |x| > r.$$

In other words, $x_n \notin N(x, r)$ if $n > |x| + r$. It follows that $N(x, r)$ can contain at most finitely many points of E (a subset of $\{x_n : n \leq |x| + r\}$). Hence x is not an accumulation point of E.

Problem 2. True or false? Justify with a brief proof or a counterexample.

- Every finite set is compact.

  TRUE: A finite set is both bounded and closed, so is compact.

- The set $\{x \in \mathbb{R} : x - x^2 > 0\}$ is compact.

  FALSE: In fact, $\{x \in \mathbb{R} : x - x^2 > 0\} = (0, 1)$, which is not closed, hence not compact.

- The set $\mathbb{Q} \cap [0, 1]$ is compact.

  FALSE: $\mathbb{Q} \cap [0, 1]$ is not closed since its closure is the interval $[0, 1]$.

- No open cover of $(-1, 1)$ can have a finite subcover.

  FALSE: The open cover $\{(-1, 1)\}$ consisting of one open set is already a finite cover.

  Note: $(0, 1)$ is not compact, so there must be *some* open cover of it with no finite subcover (such as $\{(2^{-n}, 1) : n \in \mathbb{N}\}$). It does not mean that no open cover can have a finite subcover.

Problem 3. Is it true that the union of any number of compact sets is compact? What about the union of a finite number of compact sets?

The answer to the first question is negative. For example, if $K_n = [n, n + 1]$ then each K_n is compact but $\bigcup_{n \in \mathbb{Z}} K_n = \mathbb{R}$ is not compact. However, the union of a finite number of compact sets is compact. To see this, suppose $K_1, \ldots, K_s$ are compact, so that each K_n is closed and bounded, say

$$|x| \leq M_n \quad \text{for every } x \in K_n.$$

The union $K = \bigcup_{n=1}^{s} K_n$ is closed (finite unions of closed sets are closed). Moreover, if $M = \max\{M_1, \ldots, M_s\}$, then

$$|x| \leq M \quad \text{for every } x \in K,$$

which shows K is bounded. Thus K, being bounded and closed, is compact.

Problem 4. Suppose $S \subset \mathbb{R}$ is compact. Show that $\sup(S)$ and $\inf(S)$ exist and both belong to S.

Since S is bounded above and below, $\sup(S)$ and $\inf(S)$ exist by the Completeness Axiom. Let us show that $b = \sup(S) \in S$, the proof of $\inf(S) \in S$ being similar. Assume by way of contradiction that $b \notin S$. Since S is compact, it is closed, so $\mathbb{R} \smallsetminus S$ is open. Since $b \in \mathbb{R} \smallsetminus S$ there exists a neighborhood $N(b, r)$ which is contained in $\mathbb{R} \smallsetminus S$. In particular, the interval $(b - r, b)$ is disjoint from S, so $b - \frac{r}{2}$ is an upper bound for S. This contradicts $b = \sup(S)$.