Math 320 Homework 6

Due Thursday October 23, 2003

Problem 1. True or false? Give a brief proof or a counterexample.

- If $\lim_{n\to\infty} |x_n| = 5$, then either $\lim_{n\to\infty} x_n = 5$ or $\lim_{n\to\infty} x_n = -5$.
- If $\lim_{n\to\infty} x_n = 1$, then $x_n < 2$ for all but finitely many values of n.
- If $0.9999 < x_n < 1.0001$ for all $n \geq 500$, then $\lim_{n\to\infty} x_n = 1$.

Problem 2. Guess the following limits and use the definition of limit to prove that your guess is correct.

- $\lim_{n\to\infty} \dfrac{1}{3^n}$
- $\lim_{n\to\infty} \dfrac{3n+1}{4n-1}$

Problem 3. Suppose b is an accumulation point of a non-empty set $S \subset \mathbb{R}$. Show that there is a sequence $\{x_n\}$ of points in S such that $\lim_{n\to\infty} x_n = b$.

Problem 4. Suppose $\{x_n\}$ is a sequence which converges to L. If $\{y_n\}$ is another sequence such that $|x_n - y_n| \leq 1/n^2$ for all n, show that $\{y_n\}$ converges to L also.

Problem 5. When you enter a positive number into your pocket calculator and keep pressing the "$\sqrt{\ }$" button, you see the outputs get closer and closer to 1, irrespective of your initial choice. Here is an explanation: Define a sequence $\{x_n\}$ by choosing x_1 to be any positive number, and then defining $x_{n+1} = \sqrt{x_n}$ for every $n \geq 1$.

- If $0 < x_1 \leq 1$, show that $\{x_n\}$ is increasing and bounded above, so it has a limit L. Use the relation $x_{n+1} = \sqrt{x_n}$ to show that $L = 1$.
- If $x_1 > 1$, show that $\{x_n\}$ is decreasing and bounded below. Conclude as above that $\{x_n\}$ converges to 1.