

Math 320 Homework 6 solutions

Problem 1. True or false? Give a brief proof or a counterexample.

- If $\lim_{n \rightarrow \infty} |x_n| = 5$, then either $\lim_{n \rightarrow \infty} x_n = 5$ or $\lim_{n \rightarrow \infty} x_n = -5$.
FALSE: Consider $x_n = (-1)^n 5$.
- If $\lim_{n \rightarrow \infty} x_n = 1$, then $x_n < 2$ for all but finitely many values of n .
TRUE: Since $\lim_{n \rightarrow \infty} x_n = 1$, there exists an n_0 such that $|x_n - 1| < 1$ or $0 < x_n < 2$ whenever $n \geq n_0$.
- If $0.9999 < x_n < 1.0001$ for all $n \geq 500$, then $\lim_{n \rightarrow \infty} x_n = 1$.
FALSE: Consider the constant sequence in which $x_n = 0.99995$ for all n .

Problem 2. Guess the following limits and use the definition of limit to prove that your guess is correct.

- $\lim_{n \rightarrow \infty} \frac{1}{3^n}$

We show that the limit is 0. Given $\varepsilon > 0$, let n_0 be any integer greater than

$$\frac{\log(\frac{1}{\varepsilon})}{\log 3}.$$

Then, if $n \geq n_0$, we have

$$\begin{aligned} n > \frac{\log(\frac{1}{\varepsilon})}{\log 3} &\implies n \log 3 > \log(\frac{1}{\varepsilon}) \\ &\implies \log(3^n) > \log(\frac{1}{\varepsilon}) \\ &\implies 3^n > \frac{1}{\varepsilon} \\ &\implies \frac{1}{3^n} < \varepsilon. \end{aligned}$$

- $\lim_{n \rightarrow \infty} \frac{3n+1}{4n-1}$

We prove that the limit is $3/4$. Given $\varepsilon > 0$, let n_0 be any integer greater than $1/\varepsilon$. Then, if $n \geq n_0$,

$$\left| \frac{3n+1}{4n-1} - \frac{3}{4} \right| = \left| \frac{7}{4(4n-1)} \right| = \frac{7}{4} \cdot \frac{1}{4n-1} < \frac{2}{4n-1} < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

Problem 3. Suppose b is an accumulation point of a non-empty set $S \subset \mathbb{R}$. Show that there is a sequence $\{x_n\}$ of points in S such that $\lim_{n \rightarrow \infty} x_n = b$.

We know that every neighborhood of b contains infinitely many points of S . So for every $n \in \mathbb{N}$, we can find $x_n \in S$ such that $|x_n - b| < \frac{1}{n}$. It is easy to check that $x_n \rightarrow b$: Given $\varepsilon > 0$, let n_0 be any integer greater than $1/\varepsilon$. Then, if $n \geq n_0$, we have $|x_n - b| < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$.

Problem 4. Suppose $\{x_n\}$ is a sequence which converges to L . If $\{y_n\}$ is another sequence such that $|x_n - y_n| \leq 1/n^2$ for all n , show that $\{y_n\}$ converges to L also.

Given $\varepsilon > 0$, find n_0 such that $|x_n - L| < \varepsilon$ whenever $n \geq n_0$. Let n_1 be any integer greater than $1/\sqrt{\varepsilon}$. Set $n_2 = \max\{n_0, n_1\}$. Then, if $n \geq n_2$, we have

$$|y_n - L| \leq |y_n - x_n| + |x_n - L| < \frac{1}{n^2} + \varepsilon \leq \frac{1}{n_1^2} + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon.$$

Problem 5. When you enter a positive number into your pocket calculator and keep pressing the “ $\sqrt{\quad}$ ” button, you see the outputs get closer and closer to 1, irrespective of your initial choice. Here is an explanation: Define a sequence $\{x_n\}$ by choosing x_1 to be any positive number, and then defining $x_{n+1} = \sqrt{x_n}$ for every $n \geq 1$.

- If $0 < x_1 \leq 1$, show that $\{x_n\}$ is increasing and bounded above, so it has a limit L . Use the relation $x_{n+1} = \sqrt{x_n}$ to show that $L = 1$.

We know from calculus that $x \leq \sqrt{x} \leq 1$ if $0 < x \leq 1$. Thus

$$x_1 \leq \sqrt{x_1} = x_2 \leq \sqrt{x_2} = x_3 \leq \cdots \leq 1$$

which shows $\{x_n\}$ is increasing and bounded above by 1. Hence $\lim_{n \rightarrow \infty} x_n = L$ exists. To find L , take limit of both sides of the relation $x_{n+1} = \sqrt{x_n}$ to obtain $L = \sqrt{L}$. The only solutions of this equation are $L = 0$ and $L = 1$. But $L = 0$ is not possible since $x_n \geq x_1 > 0$ for all n implies $L \geq x_1 > 0$. So $L = 1$.

- If $x_1 > 1$, show that $\{x_n\}$ is decreasing and bounded below. Conclude as above that $\{x_n\}$ converges to 1.

This is completely similar to the previous case.