Math 320 Homework 7

Due Thursday October 30, 2003

Problem 1. True or false? Justify your answer.

- If $x_n \leq y_n$ for all n and $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.
- If $x_n \neq 0$ for all n and $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} 1/x_n = +\infty$.
- The sequence $\{2^{\cos n}\}$ has a convergent subsequence.

Problem 2. (*Sandwich Lemma*) Suppose $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $\mathbb{R}$ such that $x_n \leq y_n \leq z_n$ for all n. If $\{x_n\}$ and $\{z_n\}$ both converge to the same limit L, show that $\{y_n\}$ converges to L also.

Problem 3. Show that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

(Hint: Verify that $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$ for every $n \in \mathbb{N}$ and apply the Sandwich Lemma above.)

Problem 4. Let

$$x_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

Prove that $\{x_n\}$ is not convergent by showing that it is not a Cauchy sequence. (Hint: Verify that the inequality $x_{2n} - x_n > 1$ holds for every $n > 1$.)

Problem 5. Give an example of

- an unbounded sequence in $\mathbb{R}$ which has a subsequence converging to 0.
- a sequence in $\mathbb{R}$ which has three subsequences converging to -1, 0, and 5.

Bonus problem. Show that every sequence in $\mathbb{R}$ has a monotone subsequence. In other words, if $\{x_n\}$ is a sequence in $\mathbb{R}$, show that there is a subsequence $\{x_{n_k}\}$ such that

$$x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots \leq x_{n_k} \leq \cdots$$

or

$$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots \geq x_{n_k} \geq \cdots$$