Math 320 Homework 7 solutions

Problem 1. True or false? Justify your answer.

- If $x_n \leq y_n$ for all n and $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.

  TRUE: Given $M > 0$, find n_0 such that $y_n < M$ whenever $n \geq n_0$. Since $x_n \leq y_n$, we obtain $x_n < M$ whenever $n \geq n_0$. Thus $x_n \to -\infty$.

- If $x_n \neq 0$ for all n and $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} 1/x_n = +\infty$.

  FALSE: Take $x_n = (-1)^n/n$. Then $x_n \to 0$ but $1/x_n = (-1)^n n$ which does not tend to $+\infty$.

- The sequence $\{2^{\cos n}\}$ has a convergent subsequence.

  TRUE: Since $-1 \leq \cos x \leq 1$ for all x, we have $0.5 \leq 2^{\cos n} \leq 2$ for all n. This means $\{2^{\cos n}\}$ is a bounded sequence. Hence it must have a convergent subsequence.

Problem 2. (*Sandwich Lemma*) Suppose $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $\mathbb{R}$ such that $x_n \leq y_n \leq z_n$ for all n. If $\{x_n\}$ and $\{z_n\}$ both converge to the same limit L, show that $\{y_n\}$ converges to L also.

Given $\varepsilon > 0$, find n_0 and n_1 such that

$$|x_n - L| < \varepsilon \quad \text{whenever } n \geq n_0,$$
$$|z_n - L| < \varepsilon \quad \text{whenever } n \geq n_1.$$

Set $m = \max\{n_0, n_1\}$. Then, if $n \geq m$, we have

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon.$$

In other words,

$$|y_n - L| < \varepsilon \quad \text{whenever } n \geq m,$$

which proves $y_n \to L$.

Problem 3. Show that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Note that

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n} \leq \frac{1}{n}$$

since all the fractions $\frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}$ are ≤ 1. It follows that $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$ for all n. Since $1/n \to 0$, it follows from the Sandwich Lemma that $n!/n^n \to 0$.

Problem 4. Let

$$x_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

Prove that $\{x_n\}$ is not convergent by showing that it is not a Cauchy sequence.

Note that if $n > 1$,

$$x_{2n} - x_n = \frac{1}{\sqrt{2n}} + \cdots + \frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{2n}} + \cdots + \frac{1}{\sqrt{2n}} = n \cdot \frac{1}{\sqrt{2n}} = \frac{\sqrt{n}}{\sqrt{2}} \geq 1.$$

It follows that $\{x_n\}$ is a not a Cauchy sequence (or $|x_{2n} - x_n|$ would be less than any $\varepsilon > 0$ for all large n). Hence $\{x_n\}$ is not convergent.

Problem 5. Give an example of

- an unbounded sequence in $\mathbb{R}$ which has a subsequence converging to 0.

  Let $x_n = (1+(-1)^n)n$. Then $x_{2k} = 4k$ so $\{x_n\}$ is unbounded. Also, $x_{2k-1} = 0$, so $\lim_{k\to\infty} x_{2k-1} = 0$.

- a sequence in $\mathbb{R}$ which has three subsequences converging to -1, 0, and 5.

  Define $\{x_n\}$ by

$$x_n = \begin{cases} -1 & \text{if } n = 3k \\ 0 & \text{if } n = 3k+1 \\ 5 & \text{if } n = 3k+2 \end{cases}$$

Bonus problem. Show that every sequence in $\mathbb{R}$ has a monotone subsequence. In other words, if $\{x_n\}$ is a sequence in $\mathbb{R}$, show that there is a subsequence $\{x_{n_k}\}$ such that

$$x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots \leq x_{n_k} \leq \cdots$$

or

$$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots \geq x_{n_k} \geq \cdots$$

The statement is trivial if the image of the sequence is a finite set, so assume this image is infinite. Consider three cases:

Case 1. $\{x_n\}$ is not bounded above. Set $n_1 = 1$. Since $\{x_n\}$ is not bounded above, there exists $n_2 > n_1$ such that $x_{n_1} < x_{n_2}$. Similarly, there exists $n_3 > n_2$ such that $x_{n_2} < x_{n_3}$. Continuing this way, we obtain an increasing subsequence.

Case 2. $\{x_n\}$ is not bounded below. Then by a similar argument, we can construct a decreasing subsequence.

Case 3. $\{x_n\}$ is bounded above and below. Then $\{x_n\}$ has an accumulation point $L \in \mathbb{R}$. Set $m_1 = 1$ and for each $k \geq 2$ choose x_{m_k} such that $0 < |x_{m_k} - L| < |x_{m_{k-1}} - L|$. The sequence $\{x_{m_k}\}$ must contain a subsequence $\{x_{n_k}\}$ such that $x_{n_k} > L$ for all k or $x_{n_k} < L$ for all k. In the first case $\{x_{n_k}\}$ is decreasing and in the second case $\{x_{n_k}\}$ is increasing.

Callenge. Try to do this problem only using the order on the real line without any appeal to the notion of limit or accumulation point.