Math 320 Homework 8

Due Thursday November 6, 2003

Problem 1. True or false? Justify your answer.

- If $\limsup_{n\to\infty} x_n = 2$, then $x_n > 1.999$ for all large n.
- There exists a sequence $\{x_n\}$ such that $\inf\{x_n : n \in \mathbb{N}\} = 0$ even though $\liminf_{n\to\infty} x_n = 1$.
- If f and g are defined in a neighborhood of p, and if both $f(x)$ and $f(x)g(x)$ have limits as $x \to p$, then $\lim_{x\to p} g(x)$ exists.

Problem 2. If $\{x_n : n \in \mathbb{N}\}$ is any enumeration of the rational numbers in $[0, 1]$, find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$. Justify your answers.

Problem 3. Let a and b be positive numbers. Show that

$$\lim_{n\to\infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$$

(Hint: Set $x_n = (a^n + b^n)^{1/n}$ and without losing generality assume $0 < a \leq b$. Verify that $x_n > b$ and so $\liminf_{n\to\infty} x_n \geq b$. On the other hand, verify that $x_n \leq 2^{1/n} b$ and so $\limsup_{n\to\infty} x_n \leq b$. Conclude that $\lim_{n\to\infty} x_n = b$.)

Problem 4. Using the definition of limit, verify that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

(Hint: $|\sin\theta| \leq 1$ for all θ.)

Problem 5. Using the sequential criterion for limits, verify that

$$\lim_{x\to 1} \cos\left(\frac{1}{x-1}\right)$$

does not exist.

Bonus Problem. Every rational number can be expressed uniquely as m/n, where m and n are integers with no common factor and $n > 0$. Define a function $f : \mathbb{Q} \to \mathbb{R}$ by $f(m/n) = 1/n$. Show that for every $p \in \mathbb{R}$,

$$\lim_{x\to p} f(x) = 0.$$