Math 320 Homework 8 solutions

Problem 1. True or false? Justify your answer.

- If $\limsup_{n\to\infty} x_n = 2$, then $x_n > 1.999$ for all large n.

  FALSE: Consider $x_n = (-1)^n 2$.

- There exists a sequence $\{x_n\}$ such that $\inf\{x_n : n \in \mathbb{N}\} = 0$ even though $\liminf_{n\to\infty} x_n = 1$.

  TRUE: Consider $x_n = 1 - \frac{1}{n}$.

- If f and g are defined in a neighborhood of p, and if both $f(x)$ and $f(x)g(x)$ have limits as $x \to p$, then $\lim_{x\to p} g(x)$ exists.

  FALSE: Let $p = 0$, $f(x) = x^2$, $g(x) = 1/x$. Then $f(x)$ and $f(x)g(x) = x$ have limits as $x \to 0$ but $g(x)$ does not.

Problem 2. If $\{x_n : n \in \mathbb{N}\}$ is any enumeration of the rational numbers in $[0,1]$, find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$. Justify your answers.

Since every x_n is between 0 and 1, we have $0 \le \liminf x_n \le \limsup x_n \le 1$. Since 1 is an accumulation point of $\mathbb{Q} \cap [0,1]$, there is a subsequence $\{x_{n_k}\}$ which converges to 1, hence $\limsup x_n \ge 1$. It follows that $\limsup x_n = 1$. Similarly, since 0 is an accumulation point of $\mathbb{Q} \cap [0,1]$, there is a subsequence $\{x_{m_k}\}$ which converges to 0, hence $\liminf x_n \le 0$. It follows that $\liminf x_n = 0$.

Problem 3. Let a and b be positive numbers. Show that

$$\lim_{n\to\infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$$

Set $x_n = (a^n + b^n)^{1/n}$ and without losing generality assume $0 < a \le b$. Then $x_n > (b^n)^{1/n} = b$ for every n, so $\liminf x_n \ge b$. On the other hand, $x_n \le (b^n + b^n)^{1/n} = (2b^n)^{1/n} = 2^{1/n}\, b$ for every n, so $\limsup x_n \le \limsup 2^{1/n}\, b = b \limsup 2^{1/n} = b \cdot 1 = b$. It follows that

$$b \le \liminf x_n \le \limsup x_n \le b,$$

which gives $\liminf x_n = \limsup x_n = b$, or $\lim_{n\to\infty} x_n = b$.

Problem 4. Using the definition of limit, verify that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Given $\varepsilon > 0$, set $\delta = \varepsilon$. If $|x| < \delta$, using the fact that $|\sin(\frac{1}{x})| \le 1$, we obtain

$$|x \sin\left(\frac{1}{x}\right)| \le |x| < \delta = \varepsilon.$$

Problem 5. Using the sequential criterion for limits, verify that

$$\lim_{x\to 1} \cos\left(\frac{1}{x-1}\right)$$

does not exist.

If $x_n = 1 + \frac{1}{2n\pi}$, then $x_n \to 1$ and $\cos(\frac{1}{x_n - 1}) = 1 \to 1$. On the other hand, if $y_n = 1 + \frac{1}{(2n+1)\pi}$, then $y_n \to 1$ also, but $\cos(\frac{1}{y_n - 1}) = -1 \to -1$. Since we get two different sequential limits, the limit in question does not exist.

Bonus Problem. Every rational number can be expressed uniquely as m/n, where m and n are integers with no common factor and $n > 0$. Define a function $f : \mathbb{Q} \to \mathbb{R}$ by $f(m/n) = 1/n$. Show that for every $p \in \mathbb{R}$,

$$\lim_{x\to p} f(x) = 0.$$

Fix $p \in \mathbb{R}$ and let $\varepsilon > 0$ be given. Choose a positive integer n_0 greater than $1/\varepsilon$. Consider the set E of rational numbers in the interval $(p-1, p+1)$ whose denominator is at most n_0. Note that

$$E = \left\{\frac{m}{n} : n(p-1) < m < n(p+1) \text{ and } 1 \leq n \leq n_0\right\},$$

so E is in fact a finite set. Thus, there is a $0 < \delta < 1$ such that $|x - p| \geq \delta$ whenever $x \in E$ and $x \neq p$. Now if $x = m/n \in \mathbb{Q}$ and $0 < |x - p| < \delta$, then by the above discussion $x \notin E$, so $n > n_0$. So $|f(x)| = 1/n < 1/n_0 < \varepsilon$.