Math 320 Homework 9

Due Thursday November 13, 2003

Problem 1. True or false? Justify your answer.

- If $f : D \to \mathbb{R}$ is continuous and D is a closed set, then $f(D)$ is a closed set.
- If $f : D \to \mathbb{R}$ is continuous, then $|f| : D \to \mathbb{R}$ is continuous. (Here $|f|$ is the function which takes the value $|f(x)|$ at every point $x \in D$.)
- If g and $g \circ f$ are continuous, then f is continuous.

Problem 2. Show that the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous at $p = 0$ and discontinuous at every $p \neq 0$.

Problem 3. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f(x) = g(x)$ for every $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for every $x \in \mathbb{R}$. In other words, *a continuous function on $\mathbb{R}$ is uniquely determined by its values at rational numbers.* What property of $\mathbb{Q}$ did you use in your proof?

Problem 4. Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a *fixed point*, that is, a point $p \in [0,1]$ such that $f(p) = p$. (Hint: If $f(0) = 0$ or $f(1) = 1$, there is nothing to prove. Otherwise, $f(0) > 0$ and $f(1) < 1$, and you can apply the Intermediate Value Theorem to an appropriate function.)

Problem 5. We say that a function $f : D \to \mathbb{R}$ is *two-to-one* if for every $y \in f(D)$ there are exactly two points x_1 and x_2 in D such that $f(x_1) = f(x_2) = y$. Show that there is no continuous two-to-one function $f : [0,1] \to \mathbb{R}$. (Hint: Any such function has to take its maximum M at exactly two points $a, b \in [0,1]$. We can assume for example that $a \in (0,1)$; if $a = 0$ and $b = 1$ consider the minimum instead. Then for small $\varepsilon > 0$, the equation $f(x) = M - \varepsilon$ has at least two solutions near a and one solution near b, which contradicts f being two-to-one.)