Math 320 Homework 9 solutions

Problem 1. True or false? Justify your answer.

- If $f : D \to \mathbb{R}$ is continuous and D is a closed set, then $f(D)$ is a closed set.

  FALSE: f defined by $f(x) = \frac{1}{1+x^2}$ is continuous on the closed set $D = \mathbb{R}$, yet $f(\mathbb{R}) = (0, 1]$ is not closed.

- If $f : D \to \mathbb{R}$ is continuous, then $|f| : D \to \mathbb{R}$ is continuous. (Here $|f|$ is the function which takes the value $|f(x)|$ at every point $x \in D$.)

  TRUE: Let $p \in D$ and $\varepsilon > 0$ be given. Since f is continuous at p, there is a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $x \in D$ and $|x - p| < \delta$. By triangle inequality, $||f(x)| - |f(p)|| \leq |f(x) - f(p)|$. Hence $||f(x)| - |f(p)|| < \varepsilon$ whenever $x \in D$ and $|x - p| < \delta$. This proves $|f|$ is continuous at p. Since $p \in D$ was arbitrary, it follows that $|f|$ is continuous on D.

- If g and $g \circ f$ are continuous, then f is continuous.

  FALSE: Take any discontinuous function $f : \mathbb{R} \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a constant function. Then $g \circ f$ is also constant, so g and $g \circ f$ are both continuous.

Problem 2. Show that the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous at $p = 0$ and discontinuous at every $p \neq 0$.

Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then, if $|x| < \delta = \varepsilon$, we have $|f(x) - f(0)| = |f(x)| = |x| < \varepsilon$ if x is rational and $|f(x) - f(0)| = 0 < \varepsilon$ if x is irrational. In either case, $|x| < \delta$ implies $|f(x) - f(0)| < \varepsilon$. This proves continuity at $p = 0$.

Now let $p \neq 0$. Take a sequence $\{x_n\}$ of rationals converging to p. Then $f(x_n) = x_n \to p$. Also take a sequence $\{y_n\}$ of irrationals converging to p. Then $f(y_n) = 0 \to 0$. Since $p \neq 0$, it follows that the limit of f at p does not exist. Hence f is not continuous at p.

Problem 3. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f(x) = g(x)$ for every $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for every $x \in \mathbb{R}$. In other words, *a continuous function on $\mathbb{R}$ is uniquely determined by its values at rational numbers.* What property of $\mathbb{Q}$ did you use in your proof?

Let $x \in \mathbb{R}$. If x is rational, we know that $f(x) = g(x)$. Suppose x is irrational. Take a sequence $\{x_n\}$ of rational numbers converging to x. Then, since f and g are continuous at x, we have

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x).$$

The only property of $\mathbb{Q}$ that is used in this argument is its density in $\mathbb{R}$.

Problem 4. Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a *fixed point*, that is, a point $p \in [0,1]$ such that $f(p) = p$.

If $f(0) = 0$ or $f(1) = 1$, there is nothing to prove. Otherwise, $f(0) > 0$ and $f(1) < 1$. Define $g(x) = f(x) - x$, Then g is continuous on $[0,1]$ and $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. By the Intermediate Value Theorem, we must have $g(p) = 0$ for some $p \in (0,1)$. Clearly $f(p) = p$.

Problem 5. We say that a function $f : D \to \mathbb{R}$ is *two-to-one* if for every $y \in f(D)$ there are exactly two points x_1 and x_2 in D such that $f(x_1) = f(x_2) = y$. Show that there is no continuous two-to-one function $f : [0,1] \to \mathbb{R}$.

Any such function has to take its maximum M at exactly two points $a, b \in [0,1]$ and its minimum m at two points $c, d \in [0,1]$. The four points a, b, c, d are distinct, so at least two of them are interior points. Without losing generality, assume $0 < a < b \le 1$; the other cases can be treated in a similar way. Choose points x_1, x_2, x_3 such that

$$0 < x_1 < a < x_2 < x_3 < b \le 1.$$

Then $f(x_1), f(x_2), f(x_3)$ are all less than M, so we can choose some λ such that $\max\{f(x_1), f(x_2), f(x_3)\} < \lambda < M$. By the Intermediate Value Theorem, there are $y_1 \in (x_1, a)$, $y_2 \in (a, x_2)$ and $y_3 \in (x_3, b)$ such that $f(y_1) = f(y_2) = f(y_3) = \lambda$. This contradicts f being two-to-one.