Solutions to the practice problems

- Suppose x and y are real numbers such that $|x-y|<|x|$. Show that $xy>0$.

Solution. Since $|x-y|<|x|$, we have $-|x|<x-y<|x|$ or $x-|x|<y<x+|x|$. This shows x cannot be zero (for otherwise $0<y<0$!). If $x>0$, we get $0<y<2x$, so $xy>0$. If $x<0$, we get $2x<y<0$, so again $xy>0$.

- Verify that if $0<a<2$, then $a<\sqrt{2a}<2$. Use this to show that the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

converges and find its limit.

Solution. Multiplying $a<2$ by a gives $a^2<2a$. Taking the square-root and noting that $a>0$, we obtain $a<\sqrt{2a}$. Similarly, multiplying $a<2$ by 2 gives $2a<4$ and taking the square-root we obtain $\sqrt{2a}<2$. This proves

$$\text{If } 0<a<2, \text{ then } 0<a<\sqrt{2a}<2 \qquad (*)$$

Now define a sequence $\{a_n\}$ by setting $a_1=\sqrt{2}$ and $a_n=\sqrt{2a_{n-1}}$ for every $n\geq 2$. Using $(*)$ above, it is easy to see that

$$0<a_{n-1}<a_n<2 \qquad \text{for every } n\geq 2.$$

In other words, $\{a_n\}$ is increasing and bounded above. Hence $L=\lim_{n\to\infty}a_n$ exists.

To find L, take the limit as $n\to\infty$ of both sides of the relation $a_n=\sqrt{2a_{n-1}}$ to obtain $L=\sqrt{2L}$. This gives $L^2=2L$ which has the solutions $L=0$ and $L=2$. But L cannot be zero since $a_n\geq\sqrt{2}$ for all n and so $L\geq\sqrt{2}$. Hence $L=2$.

- Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function with the property that $f(x)=0$ whenever $|x|\geq 1$. Show that for any sequence $\{x_n\}$ in $\mathbb{R}$, the sequence $\{f(x_n)\}$ has a convergent subsequence.

Solution. Since f is continuous, it must be bounded on the compact set $[-1,1]$, so there exists an $M>0$ such that

$$|f(x)|\leq M \qquad \text{for all } x\in[-1,1].$$

Since $f(x)=0$ for $|x|\geq 1$, we have in fact

$$|f(x)|\leq M \qquad \text{for all } x\in\mathbb{R}.$$

Now if $\{x_n\}$ is any sequence of real numbers, we have $|f(x_n)|\leq M$ for all n. In other words, $\{f(x_n)\}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem, $\{f(x_n)\}$ must have a convergent subsequence.

- Prove that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=x^{179}+\frac{163}{1+x^2+\sin^2 x}$$

is surjective.

Solution. First note that f is continuous everywhere, $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Given $\lambda \in \mathbb{R}$, it follows that there are real numbers $-\infty < a < b < +\infty$ such that

$$f(a) < \lambda < f(b).$$

Since f is continuous on $[a, b]$, the Intermediate Value Theorem assures that there is a $c \in (a, b)$ with $f(c) = \lambda$. Since λ was arbitrary, this shows f is surjective.

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^\beta \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $f'(x)$ exists for all x iff $\beta > 1$. Show that f' is continuous everywhere iff $\beta > 2$.

Solution. Clearly

$$f'(x) = \beta x^{\beta-1} \sin(1/x) - x^{\beta-2} \cos(1/x) \quad \text{if } x \neq 0.$$

On the other hand,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^\beta \sin(1/x)}{x} = \lim_{x\to 0} x^{\beta-1} \sin(1/x).$$

Thus, $f'(0)$ exists iff $\beta > 1$, in which case $f'(0) = 0$. So assuming $\beta > 1$, we have

$$f'(x) = \begin{cases} \beta x^{\beta-1} \sin(1/x) - x^{\beta-2} \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly f' is continuous at every $x \neq 0$. It is continuous at $x = 0$ iff $\lim_{x\to 0} f'(x) = f'(0)$ iff $\lim_{x\to 0} \beta x^{\beta-1} \sin(1/x) - x^{\beta-2} \cos(1/x) = 0$. Since $\beta > 1$, the first term on the left has limit zero. So the limit on the left is zero iff $\lim_{x\to 0} x^{\beta-2} \cos(1/x) = 0$, and this happens iff $\beta > 2$.

- Write down the Taylor polynomial $P_7(x)$ of $\sin x$ about 0. Show that $P_7(1)$ approximates $\sin(1)$ with an error of less than 10^{-4}.

Solution. By definition,

$$P_7(x) = \sum_{i=0}^{7} \frac{\sin^{(i)}(0)}{i!} x^i.$$

A brief computation gives

$$P_7(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7.$$

By Taylor's Theorem,

$$\sin(1) - P_7(1) = \frac{\sin^{(8)}(c)}{8!} \cdot 1^8 \qquad \text{for some } c \in (0, 1).$$

Since $\sin^{(8)}(x) = \sin(x)$, we obtain

$$|\sin(1) - P_7(1)| = \frac{|\sin(c)|}{8!} \leq \frac{1}{8!} \approx 0.0000248 < 10^{-4}.$$

- Discuss the integrability of the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \smallsetminus \mathbb{Q} \end{cases}$$

Solution. Let $P = \{0 = x_0, x_1, \dots, x_{n-1}, x_n = 1\}$ be any partition of $[0,1]$. Then, since both rationals and irrationals are dense in $[0,1]$,

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = 0 \quad \text{and} \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = x_i.$$

Thus,

$$L(f,P) = \sum_{i=1}^{n} m_i \, \Delta x_i = 0$$

and

$$U(f,P) = \sum_{i=1}^{n} M_i \, \Delta x_i = \sum_{i=1}^{n} x_i \, \Delta x_i.$$

The last sum is the upper sum $U(g,P)$ for the function $g(x) = x$, so

$$U(f,P) = U(g,P) \geq \int_0^1 x \, dx = \frac{1}{2}.$$

Thus, for any partition P,

$$U(f,P) - L(f,P) \geq \frac{1}{2}.$$

It follows from the basic integrability criterion that f is not integrable.

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$. Find

$$\lim_{x \to a} \frac{1}{x-a} \int_a^x f(t) \, dt.$$

Solution. Define $F(x) = \int_a^x f(t) \, dt$, so that $F(a) = 0$. By the first Fundamental Theorem of Calculus, F is differentiable at a and $F'(a) = f(a)$. Thus

$$\lim_{x \to a} \frac{\int_a^x f(t) \, dt}{x-a} = \lim_{x \to a} \frac{F(x) - F(a)}{x-a} = F'(a) = f(a).$$

- Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $A, B : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_{A(x)}^{B(x)} f(t) \, dt.$$

Prove that g is differentiable and find a formula for $g'(x)$.

Solutions. Define $F(x) = \int_0^x f(t)\,dt$. Since f is continuous, the first Fundamental Theorem of Calculus shows that F is differentiable everywhere and $F'(x) = f(x)$. Note that

$$g(x) = \int_0^{B(x)} f(t)\,dt - \int_0^{A(x)} f(t)\,dt = F(B(x)) - F(A(x)).$$

Since A, B, F are differentiable, so is g. By the chain rule,

$$g'(x) = F'(B(x))B'(x) - F'(A(x))A'(x) = f(B(x))B'(x) - f(A(x))A'(x).$$

- Show that the Mean Value Theorem for Integrals (problem 4 of homework 11) can be deduced from the Mean Value Theorem for Derivatives and the first Fundamental Theorem of Calculus.

Solution. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Define $F(x) = \int_a^x f(t)\,dt$. By the first Fundamental Theorem of Calculus, $F : [a,b] \to \mathbb{R}$ is differentiable, $F'(x) = f(x)$ for all $x \in [a,b]$ and $F(a) = 0$. By the Mean Value Theorem for Derivatives, there is $c \in (a,b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c) \qquad \text{or} \qquad \frac{F(b)}{b - a} = f(c).$$

Hence,

$$\frac{\int_a^b f(t)\,dt}{b - a} = f(c).$$