Math 328 Homework 4

due on Thursday 2/27/20

Problem 1. Let $f(x) = \sin x$ for $0 < x < \pi$.

(i) Sketch the graph of the π-periodic extension of f over a few periods. Is this a piecewise C^1 function?

(ii) Compute the Fourier series of this extension. Use this Fourier series and a relevant convergence theorem to conclude that

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx) \qquad \text{if } 0 \leq x \leq \pi$$

(iii) Evaluate both sides of the above formula at two suitable values of x to obtain the summation formulas

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} - \frac{\pi}{4}.$$

Problem 2. Let $f(x) = \pi - x$ for $0 < x < 2\pi$ and extend f as a 2π-periodic function to the real line. Sketch the graphs of $f(x)$, $f'(x)$ and $g(x) = \int_0^x f(t)\,dt$ for $-4\pi \leq x \leq 4\pi$. Without computing anything, determine whether term-by-term differentiation and integration of the Fourier series of f will give the Fourier series of f' and g.

Problem 3.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and

$$\mathrm{FS}(f)(x) = a_0 + \sum_{n=1}^{\infty} [\, a_n \cos(nx) + b_n \sin(nx) \,].$$

According to the Basic Differentiation Theorem, when f is continuous and piecewise C^1,

$$\mathrm{FS}(f')(x) = \sum_{n=1}^{\infty} [\, nb_n \cos(nx) - na_n \sin(nx) \,].$$

Can you explain *directly* the absence of the constant term in this Fourier series? (Hint: The constant term is the average value of f' over $[-\pi, \pi]$.)

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and

$$g(x) = \int_0^x f(t)\,dt.$$

Show that g is 2π-periodic if and only if $\int_0^{2\pi} f(t)\,dt = 0$.