Math 328 Homework 7

due on Friday 4/17/20

Problem 1. Solve the Laplace equation

$$\begin{cases} \Delta u = 0 & 0 < x < \pi,\ 0 < y < \pi \\ u(x,0) = u(x,\pi) = \sin(x) & 0 < x < \pi \\ u(0,y) = u(\pi,y) = \sin(y) & 0 < y < \pi \end{cases}$$

Problem 2. Solve the Laplace equation

$$\begin{cases} \Delta u = 0 & 0 < x < 2,\ 0 < y < 1 \\ u(x,0) = 3x,\ \ u(x,1) = \sin(\pi x) + 4x & 0 < x < 2 \\ u(0,y) = 0,\ \ u(2,y) = \sin(\pi y) + 2y + 6 & 0 < y < 1 \end{cases}$$

(Hint: Split the boundary conditions appropriately.)

Problem 3. Consider the Neumann problem

$$\begin{cases} \Delta u = 0 & 0 < x < L,\ 0 < y < M \\ u_y(x,0) = S(x),\ \ u_y(x,M) = N(x) & 0 < x < L \\ u_x(0,y) = W(y),\ \ u_x(L,y) = E(y) & 0 < y < M \end{cases}$$

where instead of the value of u on the boundary of the rectangle, the normal derivative of u is prescribed by the given functions $S(x), N(x), W(y), E(y)$.

(i) Show that in order for this problem to have a solution, it is necessary to have

$$\int_0^L N(x)\,dx - \int_0^L S(x)\,dx + \int_0^M E(y)\,dy - \int_0^M W(y)\,dy = 0.$$

(Hint: Write $\int_0^M \int_0^L (u_{xx} + u_{yy})\,dx\,dy = 0$ and apply the fundamental theorem of calculus (or Green's theorem) to the left.)

(ii) Suppose $S(x)$, $W(y)$, and $E(y)$ are identically zero. Use separation of variables to find the formal solution of the above problem. Note that given any solution $u(x,y)$ and any constant c, the function $u(x,y) + c$ is also a solution, which means the solution is only determined up to an additive constant.