Math 328 Homework 8

due on Friday 4/24/20

Problem 1. Find the solution of the Laplace equation

$$\begin{cases} \Delta u = 0 & 0 \le r < 1, -\pi \le \theta \le \pi \\ u(1,\theta) = \sin(3\theta) \end{cases}$$

in the unit disk given by the Fourier method (separation of variables). Then compare your answer with the one given by the Poisson integral formula to compute the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin(3t)}{5 - 4\sin t}\, dt.$$

Problem 2.

(i) Let $c > 0$ and consider the Poisson kernel

$$P(c,r,t) = \frac{c^2 - r^2}{c^2 - 2cr\cos t + r^2}$$

in the disk $r < c$. Show that

$$\frac{c-r}{c+r} \le P(c,r,t) \le \frac{c+r}{c-r}.$$

(ii) Suppose a function u is harmonic in the disk $r < c$ and continuous on the closed disk $r \le c$. Assume further that $u(r,\theta) \ge 0$. Use the Poisson integral formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(c,t)\, P(c,r,\theta - t)\, dt \qquad (r < c)$$

to prove *Harnak's inequalities*

$$\frac{c-r}{c+r}\, u(0,\theta) \le u(r,\theta) \le \frac{c+r}{c-r}\, u(0,\theta) \qquad (r < c)$$

Here $u(0,\theta)$ denotes the value of u at the origin.

Problem 3. Suppose u is a positive harmonic function which is defined everywhere in the plane. Show that u must be a constant function (Hint: Use Harnak's inequalities).