Math 328 Homework 9

due on Friday 5/1/20

Problem 1. Find the Fourier integral of the functions

$$f(x) = \begin{cases} -|x| + 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \\ 0 & |x| > 1 \end{cases}$$

For what values of x do the equalities

$$\mathrm{FI}(f)(x) = f(x) \qquad \text{and} \qquad \mathrm{FI}(g)(x) = g(x)$$

hold?

Problem 2. Use the result of problem 1 to compute the improper integrals

$$\int_0^\infty \frac{1 - \cos\omega}{\omega^2}\, d\omega \qquad \text{and} \qquad \int_0^\infty \frac{(1 - \cos\omega)\sin\omega}{\omega}\, d\omega$$

Problem 3. The solution of the one-dimensional heat equation

$$\begin{cases} u_t = k u_{xx} & -\infty < x < \infty,\ t > 0 \\ u(x, 0) = f(x) & -\infty < x < \infty \end{cases}$$

is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4kt}}\, dy.$$

Compute $u(x, t)$ when the initial condition is given by

$$f(x) = \begin{cases} \frac{1}{2\varepsilon} & |x| < \varepsilon \\ 0 & |x| > \varepsilon \end{cases}$$

where $\varepsilon > 0$ is a constant. What happens to this solution as $\varepsilon \to 0$? Interpret you answer.