A note on first order linear PDE's

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First recall the simple case

$$au_x + bu_y + cu = f(x, y), \tag{1}$$

where a, b, c are constants and f is a smooth function of x, y.

Case 1. If one of the coefficients a or b is zero, then (1) essentially reduces to a first order linear ODE with respect to one of the variables x or y. For example, if $b = 0$, then

$$au_x + cu = f$$

which can be solved by multiplying each side by the integrating factor $\mu(x) = e^{cx/a}$ and taking the anti-derivative with respect to x.

Case 2. If both a, b are non-zero, the trick is to find appropriate new coordinates (z, w) for which the equation (1) transforms into one without the u_w term, so it can be treated as *Case 1* above. To find such coordinates, note that $au_x + bu_y$ is the directional derivative of u in the direction of the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ at slope b/a. The lines parallel to $\mathbf{v}$, called the *characteristic lines* of the equation (1), are the solutions of the ODE

$$\frac{dy}{dx} = \frac{b}{a}$$

so they are of the form

$$y = \frac{b}{a}x + \text{const.} \Longrightarrow bx - ay = \text{const.}$$

If we set $w = bx - ay$, it follows that the lines $w = \text{const.}$ are parallel to $\mathbf{v}$ everywhere so $au_x + bu_y$ will be proportional to the partial derivative u_z. This suggests that we choose the new coordinates as

$$\begin{cases} z = & x \\ w = & bx - ay \end{cases} \qquad \text{with the inverse} \qquad \begin{cases} x = & z \\ y = & (bz - w)/a. \end{cases}$$

A brief computation then shows that (1) transforms into

$$au_z + cu = f(z, (bz - w)/a)$$

which can be solved as in *Case 1*.

A similar idea can be used to solve the general first order linear PDE

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y). \tag{2}$$

Here a, b, c, f are smooth functions of x, y. We look for new coordinates (z, w) which transform (2) into a simpler PDE involving only u_z. Now $a(x, y)u_x + b(x, y)u_y$ is the directional derivative of u in the direction of the vector field $\mathbf{v}(x, y) = a(x, y)\mathbf{i} + b(x, y)\mathbf{j}$

having the slope $b(x,y)/a(x,y)$ at each point (x,y). The curves that are tangent to $\mathbf{v}(x,y)$ everywhere, called the *characteristic curves* of the equation (2), are the solutions of the ODE

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{3}$$

Suppose we can solve this ODE and represent the solutions as the level sets of a function $h(x,y)$, i.e., suppose that the solutions of (3) are of the form

$$h(x,y) = \text{const.}$$

If we set $w = h(x,y)$, it follows that the curves $w = $ const. are the characteristic curves, hence are tangent to $\mathbf{v}(x,y)$ everywhere, and $a(x,y)u_x + b(x,y)u_y$ will again be proportional to the partial derivative u_z. This suggests that we choose the new coordinates as

$$\begin{cases} z &= x \\ w &= h(x,y) \end{cases} \qquad \text{with the inverse} \qquad \begin{cases} x &= z \\ y &= \hat{h}(z,w). \end{cases}$$

The PDE (2) then transforms into

$$\hat{a}(z,w)u_z + \hat{c}(z,w)u = \hat{f}(z,w), \tag{4}$$

where the new coefficients $\hat{a}(z,w) = a(z,\hat{h}(z,w)), \hat{c}(z,w) = c(z,\hat{h}(z,w)), \hat{f}(z,w) = f(z,\hat{h}(z,w))$ are obtained by substituting z for x and $\hat{h}(z,w)$ for y into the functions a, c, f. To see this, first note that since $h(x,y)$ is constant along the solutions of (3),

$$h_x dx + h_y dy = 0 \Longrightarrow \frac{dy}{dx} = -\frac{h_x}{h_y} = \frac{b}{a} \Longrightarrow ah_x + bh_y = 0.$$

Using $z_x = 1, z_y = 0, w_x = h_x, w_y = h_y$, we obtain

$$\begin{aligned} au_x + bu_y &= a(u_z z_x + u_w w_x) + b(u_z z_y + u_w w_y) \\ &= a(u_z + u_w h_x) + bu_w h_y \\ &= au_z + (ah_x + bh_y)u_w \\ &= au_z \end{aligned}$$

Substituting this into (2) then gives (4).

As a simple example, let us solve the equation $u_x + 3yu_y - 5u = 1$ subject to the condition $u(0,y) = \cos y$. The characteristic curves are the solutions to the ODE

$$\frac{dy}{dx} = 3y$$

so they have the form

$$y = \text{const.}\, e^{3x} \qquad \text{or} \qquad ye^{-3x} = \text{const.}$$

This suggests that we take

$$\begin{cases} z = & x \\ w = & ye^{-3x} \end{cases} \qquad \text{with the inverse} \qquad \begin{cases} x = & z \\ y = & we^{3z}. \end{cases}$$

The given PDE now transforms into

$$u_z - 5u = 1$$

which can be solved as an ODE with respect to z:

$$\begin{aligned} e^{-5z}u_z - 5e^{-5z}u = e^{-5z} &\Longrightarrow \frac{\partial}{\partial z}(e^{-5z}u) = e^{-5z} \\ &\Longrightarrow e^{-5z}u = \int e^{-5z}\,dz = -\frac{1}{5}e^{-5z} + K(w) \\ &\Longrightarrow u(z,w) = -\frac{1}{5} + K(w)e^{5z}, \end{aligned}$$

where K is any C^1 function. It follows that

$$u(x,y) = -\frac{1}{5} + K(ye^{-3x})e^{5x}.$$

Imposing the side condition $u(0,y) = \cos y$, we obtain

$$-\frac{1}{5} + K(y) = \cos y \Longrightarrow K(y) = \cos y + \frac{1}{5}.$$

Thus,

$$u(x,y) = -\frac{1}{5} + \left[\cos(ye^{-3x}) + \frac{1}{5}\right]e^{5x}.$$