MATH 360 Take-Home Exam II

- **Total points: 20**
- **Solutions must be written completely, carefully and clearly.**
- **You are not allowed to share your ideas or solutions with others, but you may use your notes or any book on the subject.**
- **The deadline is 12:00 pm of Friday 11/16. Hand in your exams to me at DRL 4N59, or simply slip it under the door in case I'm not there.**

Problem 1. [4 points] Let $A \subset \mathbb{R}$ be non-empty and b be an upper bound for A. Prove that $b = \sup A$ if and only if $b \in \overline{A}$.

Problem 2. [4 points] Let $\mathbb{Q}^3 \subset \mathbb{R}^3$ be the set of all vectors whose three coordinates are rational numbers. Given any $x \in \mathbb{R}^3$ and any $\varepsilon > 0$, show that there is some $y \in \mathbb{Q}^3$ such that $\|x - y\| < \varepsilon$. Here $\|\cdot\|$ is the usual distance in $\mathbb{R}^3$.

Problem 3. [4 points] Consider the real line $\mathbb{R}$ equipped with the discrete metric

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Prove that in this metric the set of integers $\mathbb{Z} \subset \mathbb{R}$ is bounded and closed but it is not compact. (This shows that the Heine-Borel theorem does not necessarily hold in every metric space.)

Problem 4. [8 points] Each of the following statements is either true or false. Make a decision in each case, and convince me that your decision is right.

(i) The set $\mathbb{Q} \subset \mathbb{R}$ of rational numbers is closed.

(ii) If $\|\cdot\|$ is any norm on the real line $\mathbb{R}$, then there is a positive constant C such that $\|x\| = C|x|$ for all $x \in \mathbb{R}$.

(iii) There exists a non-empty open set $A \subset \mathbb{R}^2$ such that $\overline{A} = \partial A$.

(iv) There are 5 compact sets $A_1, A_2, \ldots, A_5 \subset \mathbb{R}$ such that $A_1 \cup A_2 \cup \cdots \cup A_5 = \mathbb{R}$.