Math 364 first midterm solutions

Problem 1. Mark true (T) or false (F):

- For any set $A \subset \mathbb{R}$, either A or its complement $\mathbb{R} \setminus A$ is open.

  False: For example, neither $A = (0, 1]$ nor its complement $\mathbb{R} \setminus A = (-\infty, 0] \cup (1, +\infty)$ is open.

- The sphere with the equator removed is relatively open in the sphere.

  True: Let S be the unit sphere centered at the origin, E be its equator, and $X = S \setminus E$. If $p \in X$, then the z-coordinate of p is either positive or negative (it cannot be zero). Hence for small $r > 0$, the z-coordinate of all points in the ball $B(p, r)$ will be positive or negative. Hence all points in $B(p, r) \cap S$ are in fact in X. So X is relatively open in S.

- There is no continuous map $f : \mathbb{R}^3 \to \mathbb{R}$ whose image $f(\mathbb{R}^3)$ is $[0, 1] \cup [2, 3]$.

  True: Since $\mathbb{R}^3$ is path-connected, for any continuous map $f : \mathbb{R}^3 \to \mathbb{R}$, the image $f(\mathbb{R}^3)$ must be path-connected. But $[0, 1] \cup [2, 3]$ is not path-connected.

- The map $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by the formula $f(x) = 1/x$ is a homeomorphism.

  True: f is continuous as long as $x \neq 0$. It is one-to-one since $1/x = 1/x'$ implies $x = x'$. It is onto since for every $y \neq 0$, $f(1/y) = 1/(1/y) = y$. Its inverse f^{-1} is also continuous since $f^{-1}(x) = f(x) = 1/x$. So f is a homeomorphism.

- The set $X = \{(x, y) \in \mathbb{R}^2 : xy > 1\}$ is path-connected.

  False: This is clear if you draw a picture of X in the plane. To be more accurate, points $(2, 2)$ and $(-2, -2)$ are both in X but there is no path in X joining them. In fact, let $\gamma : [0, 1] \to X$ be a path with $\gamma(0) = (2, 2)$ and $\gamma(1) = (-2, -2)$, and set $\gamma(t) = (x(t), y(t))$. Note that since $\gamma(t) \in X$ for all t, we must have $x(t)y(t) > 1$. Now the function $f(t) = x(t) + y(t)$ is continuous, with $f(0) = 4$ and $f(1) = -4$. By the Intermediate Value Theorem, there must be a $t_0 \in (0, 1)$ such that $f(t_0) = 0$, But that means $x(t_0) = -y(t_0)$, or $x(t_0)y(t_0) \leq 0$, which contradicts $x(t)y(t) > 1$.

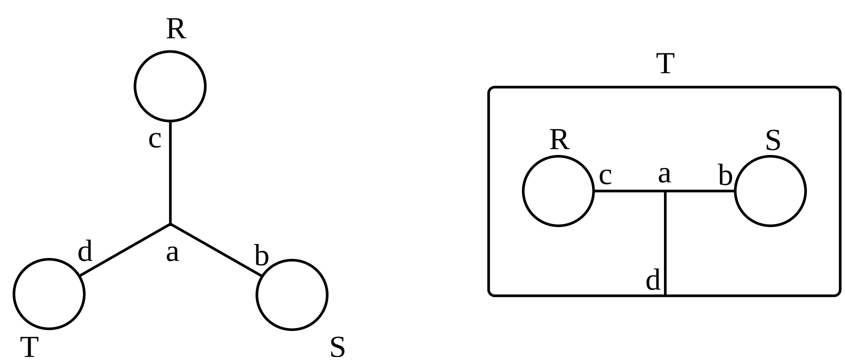
Problem 2. Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle in the plane. Carefully prove that its complement $\mathbb{R}^2 \smallsetminus C$ has two path components.

Solution. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$, so that $\mathbb{R}^2 \smallsetminus C = A \cup B$. We prove that the component of every $p \in A$ is A and the component of every $p \in B$ is B. Since every $p \in \mathbb{R}^2 \smallsetminus C$ is either in A or in B, this will prove that $\mathbb{R}^2 \smallsetminus C$ has only two components.

Fix $p \in A$. Then any point $q \in A$ can be joined to p by a straight segment which stays in $\mathbb{R}^2 \smallsetminus C$. So the component of p contains A. To see that this component is equal to A, we must show that no $q \in B$ can be joined to p by a path which stays in $\mathbb{R}^2 \smallsetminus C$. But this is clear, since if $\gamma(t) = (x(t), y(t))$ is any path joining such q to p, then $x^2(t) + y^2(t)$ is a continuous function of t whose initial value is > 1 and final value is < 1, so by IVT it must take the value 1 at some time t, meaning that the path must intersect C.

Now fix $p \in B$ and repeat the argument: Any point $q \in B$ can be joined to p by a circular arc followed by a radial segment which stays in $\mathbb{R}^2 \smallsetminus C$. So the component of p contains B. To see that this component is equal to B, we must check that no $q \in A$ can be joined to p by a path which stays in $\mathbb{R}^2 \smallsetminus C$. The argument here is identical to the previous case.

Problem 3. Are the following subsets of the plane homeomorphic? Explain why.



Solution. Yes, they are homeomorphic. On the left, label the "joints" a,b,c,d, and the round loops R,S,T, as in the picture. Define a mapping between the two figures by sending each joint on the left to the joint with the same label on the right. Then map each segment between two joints linearly to the corresponding segment. Finally map round loops R and S to their companions on the right, and map the circular loop T homeomorphically to its rectangular companion on the left.

Problem 4. Construct an explicit homeomorphism between the punctured plane $\mathbb{R}^2 \setminus (0,0)$ and the cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1\}$.

Solution. There are many ways to define such a homeomorphism; here is one. Every point $(x,y) \in \mathbb{R}^2 \setminus (0,0)$ is of the form $(r\cos\theta, r\sin\theta)$, where $r = \sqrt{x^2+y^2} > 0$ and $0 \le \theta = \arctan y/x < 2\pi$. Define $f : \mathbb{R}^2 \setminus (0,0) \to \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1\}$ by

$$f(r\cos\theta, r\sin\theta) = (\cos\theta, \sin\theta, \ln r).$$

This is easily seen to be bijective and continuous. The inverse map has the formula

$$f^{-1}(\cos\theta, \sin\theta, z) = (e^z\cos\theta, e^z\sin\theta)$$

which is also continuous, so f is a homeomorphism.

Note that you can replace $\ln r$ in the third coordinate by any homeomorphism $g : (0,+\infty) \to (-\infty,+\infty)$, such as $g(r) = r - 1/r$.