Math 364 Homework 11 (due 12/17/2002)

Problem 1. Consider any polyhedron with v vertices, e edges and f faces. Let v_n be the number of vertices of degree n, and let f_n be the number of n-gon faces. Show that

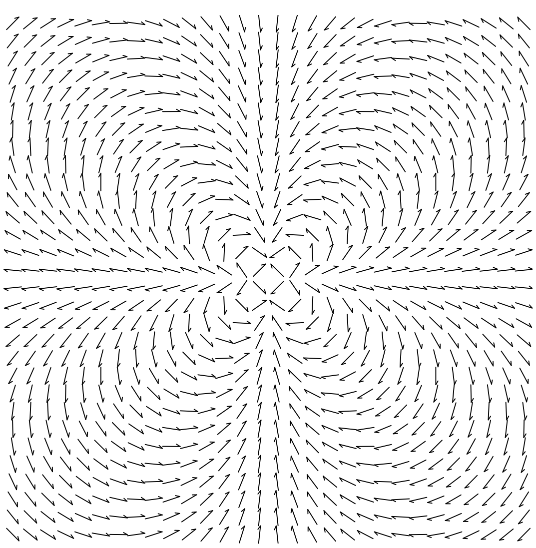
$$v_3 + v_4 + v_5 + \cdots = v \qquad f_3 + f_4 + f_5 + \cdots = f$$

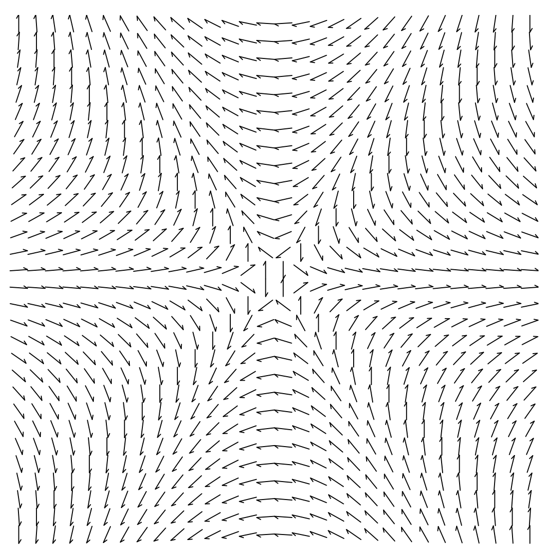
and

$$3v_3 + 4v_4 + 5v_5 + \cdots = 3f_3 + 4f_4 + 5f_5 + \cdots = 2e.$$

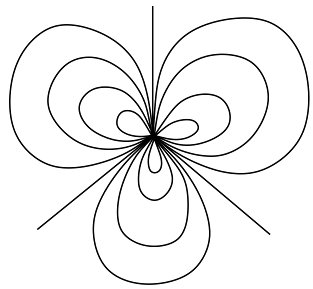
Problem 2. A standard soccer ball gives a good example of a polyhedral structure on the sphere, with 12 pentagonal and 20 hexagonal faces. Find v, e and f for this polyhedron and verify the Euler's formula $v - e + f = 2$. Find the v_n's and f_n's and verify the relations in Problem 1.

Problem 3. For each vector field, find the index of the singular point at the center of the picture:





Problem 4. Does there exist a continuous vector field in the plane whose trajectories near a singular point look like the following picture?



Problem 5. By drawing a simple diagram, show that there is a continuous vector field on the sphere with exactly two singular points of indices -1 and 3.

Problem 6. What relation do you think there is between the number of peaks, valleys, and passes on the surface of the earth?

Problem 7. (Brouwer's fixed point theorem) Let $D = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1\}$ be the closed unit disk in the plane. If $f : D \to D$ is a continuous map, then f must have a fixed point, i.e., there must be at least one point $p \in D$ such that $f(p) = p$. (Hint: If f has a fixed point on the boundary circle C, there is nothing to prove. Otherwise, the continuous vector field $V(\mathbf{x}) = f(\mathbf{x}) - \mathbf{x}$ defined on D is non-zero on the circle C and always points "inward" there. Conclude that there must be a singular point of V in the interior of D.)