Math 364 Homework 4 (due 10/3/2002)

In what follows "bijective" means one-to-one and onto. By "constructing" a homeomorphism is meant finding the explicit formula for a homeomorphism.

Problem 1. Using the topological description of continuity ("preimages of open sets are open") check that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous. (Hint: What is the preimage of an open interval $(a, b) \subset \mathbb{R}$ under f?)

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by

$$f(x,y) = \begin{cases} (x+y, y) & \text{if } y \geq 0 \\ (x,y) & \text{if } y < 0 \end{cases}$$

Show that f is bijective. What is the formula for f^{-1}? What does the image of a small ball $B(p, r)$ under f look like? (Hint: Consider separately the cases where p is above, below, or on the x-axis). Using the topological description of continuity, check that both f and f^{-1} are continuous.

Problem 3. Construct two different homeomorphisms between the intervals $[0, 1)$ and $(0, 2]$ of the real line.

Problem 4. Give an example of a bijective map $f : [0, 1] \to [0, 1]$ which is not a homeomorphism.

Problem 5. Construct a homeomorphism between the rectangles

$$X = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1 \text{ and } 0 \leq y \leq 2\}$$
$$Y = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2 \text{ and } 0 < y \leq 3\}.$$

Problem 6. Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle in the plane. Define $f : [0, 2\pi) \to C$ by $f(t) = (\cos t, \sin t)$. Show that f is bijective and continuous, but the inverse map $f^{-1} : C \to [0, 2\pi)$ is not continuous at the point $(1, 0) \in C$.

Problem 7. Let $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z \neq \pm 1\}$ (the surface of the unit sphere with the north and south poles removed) and $Y = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } -1 < z < 1\}$ (the surface of a vertical cylinder of radius 1 and height 2). Construct an explicit homeomorphism $f : X \to Y$.